MATHEMATICAL COMPETENCIES AND AWARENESS IN A TEACHER EDUCATION PRACTICE

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"How do we facilitate learning of mathematics in our teacher education programmes and relate this to mathematics learning in classrooms?" is far from a trivial question. This paper will focus on student teachers' learning of mathematical competencies through practice participation. After outlining related theoretical works, an example is used to illustrate the largely implicit facilitation of certain competencies.

Introduction

Working in mathematics education implies considering what is meant by mathematics and what we want learners to learn. The motives for learning (conceptual understanding, skills, mathematising, applying, …) determine the balance between classroom activities. My purpose here is not to discuss what the appropriate balance may be, but to explore what happens when learning to 'do mathematics' is an essential motive in a mathematics teacher education programme, and how this learning may be facilitated.

Theoretical Framework

Hans Freudenthal asserted that mathematics education should first and foremost be about mathematics as a human activity, not as a study of existing mathematical structures. He stressed discovering and organising in an interplay of content and form, mathematising, abstracting, schematising, formalising, algorithmising, verbalising, and so forth (Freudenthal 1991: 15 and 49).

A recent Danish initiative attempts to describe mathematics education outcomes across schooling levels in terms of *competencies*, including reasoning, representational competencies, symbolic and formalism competencies, modelling, communication competencies and competencies in tool use (Niss 1999).

Michael de Villiers (forthcoming) discusses the inter-relations of deductive reasoning and 'quasi-empirical methods'. He claims that the latter are vital in providing students with an understanding of what is involved in doing mathematics, in motivating and in developing mathematical 'intuition'. He focuses on proving, and thus stresses conjecturing, verification, global and heuristic refutation, and the role of proof and proving in developing understanding, though he also mentions the role of experimenting and reflecting.

John Mason, amongst others, has considered aspects such as generalising versus specialising, abstracting versus instantiating, etc. (2000: 102).

As can be seen from these short descriptions, a focus on ‘doing mathematics’ does not imply shared perceptions of what counts as knowledge, of what matters in learning mathematics, and of how the desired learning takes place. Social learning theories, socialisation theories, or activity theory can offer perspectives on how this learning occurs. Here, I work from the perspective of the social theory of learning of Etienne Wenger and others (Wenger 1998), which stressed learning as social participation in practices. This perspective recognises the historical and social context which structures and gives meaning to our activities/practices. It recognises that participation in practices shapes what we do, who we are, and how we interpret what we do. Therefore, it includes both the explicit and the tacit, such as underlying assumptions, values and shared world views. It stresses that communities and organisations are also learning: through actions and interactions, learning reproduces as well as transforms the social structures in which it takes place.

In order to learn how to generate conjectures, proofs and definitions, to critique conjectures and look for counter-examples, generalize and symbolize, the learners likely need to take part in a practice where such activities are prevalent and valued. The tacit components of this practice – its ‘common sense’ – is worked out through mutual engagement in the practice.

Charlotte Krog Skott’s research on tertiary mathematics education contributes to this perspective in relation to mathematics education. She develops a concept of *potential co-learnings*: “*interactively established possibilities or potentials for learning, which are mainly not communicated explicitly in a given teaching situation*” (2003: 13). This proved a useful concept in identifying implicit parts of a mathematics course. She found six categories of potential co-learnings to cover the implicit parts she identified: “*What are valuable mathematical activities? What are mathematical aesthetics? What are interesting mathematical questions? What are mathematical proofs? What are preliminary intuitive mathematical concepts? What are mathematical tools?*” (pp. 14-15)

These perspectives provide a framework for considering a teacher education class where the teacher has as an objective to engage the students in mathematical activity. *What characterise the practice in which these students participate, and what is the teacher’s role in promoting their participation? Which mathematical competencies do students develop and what facilitates this? Are potential co-learnings generated, and if so, what facilitates this?*

Method

The data discussed here are part of a larger research and development project, aimed at unveiling relations between experienced mathematics teacher educators’ personal theories and what they do (Jørgensen and Geldmann 1999).

Danish education of teachers for grade 0-9, at the time of the observations, consisted of two general years and two specialisation years (two majors). A class of third year mathematics student teachers was observed throughout a semester. Their mathematics lessons were recorded, and the recordings as well as various writings by the teachers were discussed in the research team. The teacher, Anna Jørgensen, was part of the research team.

The practice in the college classroom is distinct from the practice of teachers in schools. The research assumes the teacher educator as representative of a community of practice of teachers. Through this representation, the teacher brings to the classroom "the concerns, sense of purpose, identification, and emotion of participation" of the practice of mathematics teachers (Wenger 1998: 276). This grants her a strong voice in determining what is valued, what counts as competence, as knowledge, and so forth. The research assumes that many of these aspects remain tacit, but to some extent can be brought to the foreground by questioning/reflecting on practice. For this purpose, selected recordings from the classroom were discussed in detail in the research team. Thus, these discussions constitute both joint analysis and data for the research. Another source of data was interviews with students.

Observations and Analysis

The following observations are from a particular theme titled 'Generalisation' but illustrate general points. Due to space limitations, only the essence of the analysis is presented here. For more details, see (Christiansen forthcoming).

Directed by a worksheet called 'Build two towers of equal heights', the students' task was to investigate when it is possible to build two towers of equal heights with a set of rods of lengths $1,2,3,\ldots,n$. The first questions addressed instances with the 10 rods which exist in a set, moving onto hypothetical rods with length 11cm, 12cm, 13cm and $n$cm. The students were asked to find a pattern or rule, and whether their explanation was a proof. They had to consider if it is sufficient that the sum $1+2+3+\ldots+n$ is even. After sharing findings with fellow students, they were to consider the topic in the light of a number of issues related to official and unofficial discourses on mathematics education in schools.

The worksheet did not specify *how* to build the towers. The students decided to limit their investigations to 1 cm wide towers. In that sense, they engaged in delimitation of the problem, which is part of the problem handling competency (Niss 1999) and exemplifies what John Mason considers a recurrent theme in mathematics, namely freedom versus delimitation (2000). The openness of the task formulation lead students to engage with these issues (cf. Christiansen, forthcoming), and thus also contains the potential co-learnings of what constitute valuable mathematical activities and interesting questions.

PROVING

Next, some of the students started to 'play' with the rods. Others used a more theoretical approach. All groups concluded that if the sum of the lengths of the rods is an even number, two towers of equal heights can be built.

The students' conclusion rested on an abstraction from dealing with rods to working with their lengths as numbers. In doing so, the students showed that an even sum is a *necessary* condition, but not that it is possible to construct the actual towers without breaking the blocks. The teacher was aware, in the situation, that this could be used to touch on the topic of necessary versus sufficient conditions (Jørgensen 2000). She followed up on the class:

T: *Now,* we have all the time concentrated on whether... it has to be even, for the odd that definitely doesn't work. That is how we, that is how the talking has been… The topic of my question is about, can one be *certain* that if the sum is even then it'll work,

With her formulation, she embraced the students' conclusion as partially valid, but also treated it as a conjecture. She recognized their 'quasi-empirical' work, yet pushed them in the direction of deductive reasoning/proving ("be certain"). She specified her point with examples. A student viewing the problem as related to numbers only was confronted with the physical rods:

T: But I can divide my number by two, but can I be sure that I can, that I do not have to break the rods? [7 seconds of silence. The teacher looks towards student and smiles]

The teacher represents the community of practice of which the students were working towards becoming members. With this non-trivial question, the teacher opened forms of mutual engagement, which invited the students to participate in the practice she represents. It was also a challenge of the students' 'knowledge', and this invited the students to engage in negotiation of meaning (cf. Wenger 1998: 53). Through this challenge, the teacher simultaneously provided proposals around which to organize the negotiation of meaning (the teacher herself calls these *bearing marks*). The teacher captures all these aspects of how she promotes participation, when she states that her main task in the beginning of a new year is to create "a space where the students dare learning and find it worthwhile" (Jørgensen 2000: 1).

The students had to accept the invitation and the challenge, and the teacher kept on asking questions until this happened. The break-through came when a student admitted that he was unsure.

The obvious potential co-learning of what counts as an interesting mathematical question was established through the students' participation in responding to the challenge. The students' discussions turned out also to involve negotiation of perceptions of what mathematics is (Christiansen forthcoming).

The students began developing a proof. In the first stages, this took place in interaction with the teacher. The students offered suggestions to inform the verification of the conjecture, and the teacher challenged these, questioned the certainty, acknowledged what she could accept as given or shown, *but* she did *not* show the students a proof (and there are several), and she did *not* indicate how to construct a proof. Unlike the demonstrations and lectures which prevail in our classrooms, this can again be recognized as an invitation to participate in a practice, as well as the provision of 'bearing marks' which directed the students to engage in a mathematical practice of verifying, refuting, and clarifying. In that sense, the teacher's responses constituted an important, yet not the only, structuring resource around the learning process.

A potential co-learning in this phase was 'what are mathematical proofs'. The students also engaged the interplay between deductive reasoning and 'quasi-empirical methods' (de Villiers) and the mathematical reasoning competency described by Niss. The latter includes evaluating a chain of arguments, knowing what is special about mathematical proofs, knowing when a chain of arguments is a proof or could become one, being able to think up and construct chains of arguments and develop them into proofs, amongst others.

When the students and the teacher discussed whether something was a proof, or perhaps could be developed into one, they indirectly negotiated the meaning they attach to proofs and proving. The understanding of how to argue convincingly for something never was formulated in words. Yet, through the mutual engagement, it was recognized as a competency which is highly valued in the practice, and which is learned through participation.

The classroom situation illustrates the role of the teacher in establishing a practice which involves these competencies and co-learnings, in particular the importance of mutual engagement and challenging students' 'knowledge'.

GENERALISING, SYMBOLISING, VISUALISING

The students found a pattern for when two towers of equal length can be built: '*cannot, cannot, can, can*'. They queried the connection between this and their conclusion about the necessity of an even sum. This led a group to look for a formula with which to determine the numerical value of $S = \sum_{i=1}^{n} i$ for a given n.

They did so by finding the area of a 'triangle' constructed of the rods aligned in order of magnitude. In dialogue between a student at the board, the teacher and the rest of the class, two formulae were developed:

$$S = \frac{n^2}{2} + \frac{n}{2} \qquad \text{and} \qquad S = \frac{n(n+1)}{2}$$

The students convinced themselves that the two expressions are equal.

In our conversations, the teacher mentioned the importance of letting the students learn to use variables through meaningful processes, gradually developing competencies and familiarity. She referred to the above mentioned situation as an example. In that sense, the situation reflects a potential co-learning about what mathematical tools are and how to use them. Symbols make up one such tool. The students were engaged in a practice involving a number of competencies, both concerning representations and translating into symbols, and the teacher ensures the space for this by opening forms of mutual engagement and encouraging negotiation of meaning. From the perspective of social learning theory, we can say that the students were all the time involved in learning what it means to communicate mathematically:

> Learning mathematics or learning to think mathematically is learning to speak mathematically. What constitutes an acceptable grammatical construction, in mathematics, is what is approved of within the discourse. Over time, studies of the development and increasing sophistication of students' language in mathematics indicate their becoming mathematical. (Lerman, forthcoming)

In this situation, the teacher gave ample space and time for the students to explore the connection between their two conclusions. She ensured that all the students followed the explanation by the student at the board. She gently guided the process on. She maintained the focus. She asked for links between the visualizations and the symbols. Though the students' focus was on clarifying a particular issue, the teacher's actions both assisted them and gave bearing marks for what is appropriate in this type of practice, thereby establishing a number of potential co-learnings.

Discussion

It is easy to see the special language, symbols, images, etc. which prevail in this classroom. However, the analysis also indicates the evolving of conventions around what qualifies as mathematical activity. This includes the presence of several of the competencies suggested by Niss and of the potential co-learnings discussed by Krog Skott. As the teacher guides the students with her bearing marks, the co-learnings and competencies are both assimilated and negotiated. The students are acquiring identities of participation, which will inform (but not determine) their identity as *mathematics* teachers.

The worksheet introduced a task which offered students the opportunity to work with generalizations, conjecturing, proving, symbolizing, representing, problem handling, and so forth. However, the worksheet did not determine the activities. It offered opportunities for engagement, where the students could contribute in a variety of ways, contributing to the activities and engaging with others around the activities in ways which were meaningful to them, and thus allowed for building of identities within the evolving community.

It was the way in which the teacher interacted with the students around their work, which meant that the situation came to contain potential co-learnings and possibilities for engaging with various competencies. “Teaching must be opportunistic because it cannot control its own effects” (Wenger 1998: 267). The teacher must create possibilities or use existing or evolving possibilities to further the students’ mathematical learning; both in the sense of becoming familiar with accepted statements and in the sense of potential co-learnings or familiarity with the mathematical practice and thus increasing ability to communicate mathematically. This requires that the teacher is capable of noting when students’ actions and communication contains elements which can inform or be utilized in a mathematical practice. She must try to formulate challenges which contain the invitation to students to engage in mathematical activity. She must have mathematical awareness which informs her bearing marks in the planning and particular in the ‘en route thinking’ stages. She must use that mathematical awareness to see when a task or a student’s actions contain possibilities to orient the situation in the direction of the bearing marks.

On the students’ part, this requires taking some level of responsibility for their learning, ‘daring learning’ enough take part in the practice while at the same time acquiring the identity of participation needed in order to learn. Wenger talks about as “almost a theorem of love that we can open our practices and communities to others, invite them into our own identities of participation, let them be what they are not, and thus start what cannot be started” (1998: 277). This teacher is very aware of her role in creating this space for daring learning, for being what they are not. So much that this is where she puts her attention:

> While I am in the situation, I am not thinking a lot of theoretical thoughts about that right now Jeppe is creating his own learning process both of mathematics and mathematics education. Neither do I think that here the dialogue is important, the dialogue which seems to bring out the best in the other. Nor do I simply think that this thing about bending [the lined up rods] together so they fit into each other can be developed – by Jeppe or someone else – into a proof for the sum of the first n numbers. But I have all of these things in me as a readiness. They take care of themselves, all the while I am thinking that if only I am able to keep the entire group’s concentration on Jeppe’s narrative. I try ‘to give my energy’ to his long pauses, his hesitation, he searches for words for what he is in the process of finding out. What I do and think is directed by an intuitive sense of the most important in my task: to create the space which makes it so, that Jeppe and the others also next time dare the vulnerability which comes with the learning process. (Jørgensen 2000: 8)

It is humbling to watch this teacher handle the balances of the situation in a way which goes far beyond the mathematical aspects of the practice (cf. Christiansen et al. 2000). It illustrates the complexity of the practice in which the teacher invites these students to participate through opening forms of mutual engagement, and it illustrates the awareness on mathematical, psychological, sociological, and cultural aspects of the situation it requires of the teacher.

It further illustrates that while we need our teachers equipped with a readiness on all of these aspects, most of all we need them to have this sense of when something happens which can carry the learning forward for the entire class, and what it takes to maintain this ‘space’. Neither these ‘readinesses’, nor this ‘sense’ can be taught by explicit means. Our best bet so far is this kind of practice, possibly with some reflective aspects as well. Still, while I hope to have illustrated the strengths of this type of practice, we are a long way from knowing and being able to describe what it takes to make such teachers.

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