Improving Student Teachers' Mathematical Knowledge

Solange Amorim Amato
Universidade de Brasília
<sraamato@unb.br>

An action research study was performed with the main aim of investigating ways of improving STs' understanding of mathematics. The teaching strategies used to improve STs' understanding were similar to the ones suggested for their future use in teaching children. The data collected indicated that most STs improved their understanding, but some STs needed more time to re-learn certain content in the primary school curriculum. This paper presents some practical solutions proposed to ameliorate the problem within the time available.

This study arises from my experiences in researching my own teaching in pre-service and in-service teacher education courses in Brazil. It was apparent that many primary school teachers and student teachers had not an appropriate understanding of the mathematics they were supposed to teach. For Skemp (1976) relational understanding involves knowing both what to do and why it works, while instrumental understanding involves knowing only what to do, the rule, but not the reason why the rule works. It is widely recognised that primary school teachers and student teachers (*STs*)' understanding of mathematics is mainly instrumental (e.g., Ball and McDiarmid, 1990; Simon, 1993; Ma, 1999; and Goulding et al., 2002). To Carré and Ernest (1993), subject-matter knowledge (*SMK*) shapes the way teachers teach and it is too important for teaching "to be acquired incidentally through classroom experiences" (p. 50). Ball and McDiarmid (1990) argue that continued documentation about teachers and STs' lack of SMK will not contribute much to ameliorate the problems encountered in teacher education and teaching. The implication is that research-based methods of tackling the problem are required.

Bennett (1993) recommends that primary school teachers should have the necessary subject knowledge for teaching mathematics to the highest level expected of children doing that stage of schooling. Teachers should have relational understanding also at a reflective and formal level (Thompson, 1985). According to Ball (1990), this "includes the ability to talk about and model concepts and procedures" (p. 458). Simon (1993) recommends that the mathematics education of primary school STs should focus more in helping them to understand, construct connections and organise the knowledge they will have to teach than in teaching additional content. Sowder et al. (1993) describe the goals and principles they have adopted in re-teaching rational numbers to STs. Among other things, they suggest that teacher educators should provide STs with (a) opportunities to confront any misconceptions, (b) experiences involving a variety of representations for rational numbers, and (c) considerably more class time to develop rational number sense.

Some teacher educators also suggest that the integration between the teaching of mathematical content and pedagogy is beneficial to teachers' and STs' acquisition of relational understanding of mathematics (e.g., Stoddart et al., 1993 and Ball and Bass, 2000). They often associate such integration with re-teaching mathematics to teachers and STs by using the same methods that can be used to teach mathematics in a relational way to school children. According to Stoddart et al. (1993), teachers tend to teach in the way they were taught. STs re-learning of mathematics through children' activities may also be a way of avoiding the future reproduction of traditional ways of teaching based on rote learning. Weissglass (1983) proposes that STs should first experience in practice the pedagogical theories which will be formalised in the future. Brown (1992) argues along similar lines and sees some similarity between the acquisition of mathematical knowledge and the acquisition of pedagogical knowledge. According to her, theory is constructed by the learners “by becoming aware of their own actions” (p. 39). The idea that action precedes theorisation was particularly important for the present study which involved helping STs to develop simultaneously both types of theory (mathematics content and pedagogy).

In the literature about STs’ SMK, there are some results of teacher educators efforts to improve STs’ mathematical knowledge. Graeber and Tirosh (1988) mention two small scale, short-term interventions in order to help STs overcome some of their misconceptions related to division. One intervention involved the use of conflict teaching in individual interviews. The second intervention involved the use of two interactive computer programs to help STs overcome their difficulties with division word problems with a dividend smaller than the divisor. Tirosh and Graeber (1990) also used conflict teaching as a means of probing STs’ misconceptions that in a division the quotient must be less than the dividend. Tierney et al. (1990) attempted to change STs’ misconceptions about area by providing them with similar paper activities to the ones used for teaching children. These were mainly initial activities for finding areas without formulas. Simon and Blume (1994) used problem solving, group work and whole-class discussions to help STs improve their knowledge of ratio and of the relationship between area and multiplication. Stoddart et al. (1993) used concrete materials and iconic representations in an attempt to reconstruct STs’ understanding of rational number concepts. The studies reviewed in this paper were concerned with the re-teaching of particular mathematical contents. More long-term interventions involving the re-teaching of several contents in the primary school curriculum are needed.

Methodology

I carried out an action research (Amato, 2001) with the aims of improving primary school STs’ understanding of, and attitudes to, mathematics. Action research is said to be “the research method of preference whenever a social practice is the focus of research activity” (Carr and Kemmis, 1986, p. 165). The main research problem, that is, primary school teachers and STs’ instrumental understanding of mathematics, was identified through my experiences in working with pre-service and in-service courses in teacher education. This research was, therefore, seen as very much related to a social practice. I had already been a practitioner in some of the possible fields of action and had a desire to try out some ideas in order to attempt to change STs' understanding of mathematics.

The study was performed at University of Brasília, Brazil, through a mathematics teaching course component (*MTCC*) in pre-service teacher education. The component consists of one semester (80 hours) in which both theory related to the teaching of mathematics and strategies for teaching the content in the primary school curriculum must be discussed. This is the only compulsory course component related to mathematics offered to primary school STs. There is an optional course component about mathematics teaching (*MTCC2*), but it is offered only very rarely because of shortage of mathematics teachers. There were two main action steps and each had the duration of one semester thus each action step took place with a different cohort of STs. A new teaching program was designed in an attempt to: (a) improve STs’ relational understanding of the content they would be expected to teach in the future and (b) improve their liking for mathematics. Four data collection instruments were used to monitor the effects of the strategic actions: (a) diary, (b) questionnaires, (c) interviews, and (d) pre- and post-tests. Much information was produced by these instruments but, because of the limitations of space, only some STs' responses to the questionnaires and interviews are reported.

In the action steps of the research the re-teaching of mathematics was integrated with the teaching of pedagogical content knowledge (*PCK*) (Wilson et al., 1987) by asking the STs to perform children's activities which have the potential to develop relational understanding of the subject. The children's activities performed by the STs had four more specific aims in mind: (a) promote STs' familiarity with multiple modes of representation for most concepts and operations in the primary school curriculum, (b) expose STs to several ways of performing operations with concrete materials, (c) help STs to construct relationships among concepts and operations through the use of versatile representations, and (d) facilitate STs' transition from concrete to symbolic mathematics. Versatile representations like straws, part-whole diagrams, and number lines (English and Halford, 1995) were often used in the practical and written activities for arithmetic in an attempt to help STs relate natural numbers to fractions and decimals. The idea is to represent together two or more related concepts in order to make their relationships clear (e.g., 35 whole straws and 3 pieces of ¼ to represent the mixed number 35¾).

In my previous courses activities involving translations among and within multiple modes of representation and the use of versatile representations were being advocated to help children construct relationships among mathematical concepts and operations, but they were not being used with the teachers and STs often enough. As STs needed to improve their relational understanding of the content in the primary school curriculum, they also needed to be treated as learners of mathematics. About 90% of the new teaching program became children's activities. I think that a strong relational understanding of mathematics (the whole) involves knowing much its content (the parts) and how the content has been put together (the connections). As the STs could be teaching any primary school grade in a very near future, they needed to perceive the curriculum as a more coherent and organic whole. Therefore, I decided to help them acquire some relational understanding of a good range of mathematical content they were supposed to teach. Teaching time was anticipated to be the greatest problem in this research as all teaching strategies I know to help children acquire relational understanding require a good amount of time to be put into practice. An analysis of the MTCC syllabus was performed in order to reduce or exclude certain items and increase the time devoted to the teaching of more complex mathematical content. I decided to: (a) reduce the theoretical content of the syllabus and (b) exclude from the program a few mathematical contents such as small numbers (zero to 9) and measurement of capacity, mass and time.

Results

The primary school STs involved in the action steps of this research were assumed to have enough instrumental understanding because they had to study mathematics during 11 years and revise all school mathematics to do the entrance examination at University of Brasília. They were mainly female from different years in the teacher education course and their ages and liking for mathematics varied. Some STs had previously done a vocational teacher education course at school level and were already qualified as primary school teachers. They were seeking a second qualification at university level. There were also a few STs from other departments for whom the MTCC was not compulsory. The results presented in this paper are mainly concerned with the first two action steps (the first and the second semesters) of this research.

Only a few STs presented some relational understanding in the pre-tests, mainly concerning addition and subtraction of natural numbers and a few fraction concepts. Most of them were STs who had done the vocational course. The pre-test median mark was 1 and the post-test median mark was 7. The difference in the two medians indicates a considerable improvement in understanding, as judged by the tests. Yet one of my main worries related to the program was the effect on the STs of asking them as adults to perform many children's activities along one whole semester. The data tended to show that most of the STs did not mind experiencing children's activities. They appeared to accept it as a normal strategy in a course component about teaching children. Many STs mentioned that experiencing children's activities had been a positive aspect of the program and had improved their understanding of mathematics. For example, "[questionnaire, first semester ST] To experience the activities is very positive, as many times the teacher teaches the content to children without having understood it him/herself" or "[questionnaire, second semester ST] The way mathematics was presented, through concrete materials and the relaxed way, led us to conclusions not previously understood".

Having said all that, it does not mean that there were not problems connected to the idea of using children' activities with STs. A ST commented about the adult thinking being different from the child's thinking: "I liked the pleasure of working with the concrete materials, keeping in mind their function for child's learning. However, although we try to pretend, the adult's thinking can't be exactly the same as the child's thinking". Indeed the STs' thinking, when performing children's activities, was affected by their previous and long experiences in learning instrumental mathematics at school. Although the selection of representations to be used by the STs in the children's activities was based on two pedagogical criteria: (a) clear embodiment of concepts and (b) versatility, some STs also presented difficulties in learning about multiple representations for certain mathematics contents. The most difficult contents for the STs were: (a) multiplication by two-digit numbers (e.g., 23x148), (b) the distinction between the sharing and the measurement interpretations of division, and (c) all about operations with fractions and decimals. Another problem was the number of STs enrolled in each class (42 in the first semester and 44 STs in the second semester). Several STs complained about the class size and explained that the number of STs did not allow me to provide the necessary amount of individual attention.

The first action step

The sequence of children' activities in the teaching program was designed so that STs always experienced practical work with concrete materials or measurement instruments before experiencing activities with iconic representations and activities with only symbols. However, the STs knew all the symbols in the primary school curriculum and had already memorised most of the symbolic algorithms they were asked to perform with concrete materials. They were, in fact, being asked to translate from symbolic to concrete representations as they already had instrumental understanding of the mathematical content in the program. During a whole class discussion about how to use concrete materials to find a common denominator to add two fractions, Carlos commented: "The child will be learning like that but we are unlearning and re-learning. Everything is too abstract at school". Then Maria continued by saying: "We are so used to doing things in the abstract that now we are having problems in visualising them in the concrete". In the interview Daniela mentioned that she was having problems in re-learning mathematics with the use of concrete materials and explained: “I have already everything in my head. Perhaps what I have in my head is easier. I do not have to do much, there is no loss of time”.

These STs had memorised symbolic ways of performing the operations which seemed to be interfering with their understanding of more informal representations for these operations. Leandro said that the procedure for dividing two fractions kept coming to surface whenever he was performing division of fractions with concrete materials. On the other hand, Angela thought she did not have any problems in learning any of the operations with fractions using concrete materials because she had forgotten how to do them with symbols: “I did not remember that I had to multiply by the inverse fraction in division. It was like learning division that day. The things that I had forgotten with symbols I could learn more easily with the concrete materials”. At the end of the first semester Angela said that she had not had any problems with most of the representations used in the teaching program. The only exception had been the understanding of the area representation for multiplication by two-digit numbers (Sugarman and Steward, 1994). She thought that her difficulties were related to the way she had memorised the algorithm by rote making no relationships with the place value of the digits. The sum 38 x 47 was verbalised as “8 x 7, 8 x 4, jump a place, 3 x 7 and 3 x 4. I did not interpret the three in the tens’ place as 30 times and the four as 4 tens or 40”.

Both theory and research results tend to support the idea that relational understanding should precede symbolism and automaticity of procedures (e.g., Hiebert and Carpenter, 1992 and Chinn and Ashcroft, 1993). The use of concrete materials and iconic representations are more profitable in the earlier stages of concept acquisition, by providing a useful starting-point for the development of relational understanding and abstract thinking. Hiebert and Carpenter (1992) suggest that after achieving automaticity learners become more reluctant to connect their symbolic procedures to other mathematical representations that could provide further links to relational knowledge. They relate this to a phenomenon called *functional fixedness* by Gestalt psychologists. The steps in a procedure may become tightly connected and fixed in the learner’s mind, not allowing a more flexible way of thinking about them. Besides concrete materials and iconic representations carry with them interpretation difficulties inherent in a representational system. They involve visual conventions and, if the learner does not know the conventions, (s)he cannot interpret them (e.g., Shuard and Rothery, 1984 and Hiebert and Carpenter, 1992). In order to help in the construction of relationships any type of representation needs to be become familiar to the learner.

Decisions made: Primary school teachers have the social responsibility of helping children learn mathematics. They must develop the ability to work backwards from their symbolic ways of representing mathematics to more informal ways of representing the subject (Ball and Bass, 2000). Representations that proved to be difficult for some STs, like the area representation for multiplication by two-digit numbers, are considered to be useful in helping students understand the multiplication algorithm (e.g., Thompson, 1999) and in teaching dyslexic students (Chinn and Ashcroft, 1993). It was more appropriate to look for ways of helping STs learn and be fluent in using these representations than excluding them from the program. The activities for difficult representations and content were started earlier in the second semester. More children’s activities were planned for the content that proved to be more difficult for STs. For this reason, the number of children’s activities for place value and addition and subtraction of natural numbers was reduced in the second semester.

The second action step

Although the teaching program was improved from the first to the second action step, a few second semester STs also presented some difficulties with certain representations and content. Each new activity that has the potential for developing relational understanding contribute to make the mathematical content clearer for the STs and to enlarge the related schemas. Some STs only needed a few activities to understand the content and construct relationships while others needed more activities. Mariana said that the teaching pace during addition and subtraction of fractions was fast for her. Alice said in the interview that she needed to re-learn fractions in a similar pace provided for child learning: "I started to understand much more about fractions. However, I have difficulties and I need much more, to the same extent as a child". Many STs suggested increasing the teaching time for operations with rational numbers because fractions and decimals were much more difficult for them than operations with natural numbers. Some STs also suggested increasing the number of activities for geometry and measurement because they were very enjoyable and they had not had enough experiences with these contents as school students.

Decisions made: The STs who were having more difficulties were advised to seek the extra help provided by me and my two teaching assistants. Individual teaching or in small groups was thought to be more appropriate in the case of these STs. The number of children's activities for representations and content that proved to be more difficult for STs in previous semesters was further increased in the third and subsequent semesters. The activities for rational numbers and geometry and measurement were started at the first week of the semester and they continued until the last day of each semester. The idea was to provide STs with several opportunities for revising past content through activities involving extensions of the content and relationships with other contents. The number of activities for operations with natural numbers alone was greatly reduced, but there were still many activities about operations with rational numbers which included a natural number part. Through operations with mixed numbers and decimals (e.g., 35¾+26¼ or 24.75-12.53) with the use of versatile representations, STs experienced further activities related to operations with natural numbers and had the opportunity to make important relationships between operations with natural numbers and operations with fractions and decimals. These changes proved to be effective in helping other classes of STs overcome their difficulties in relearning mathematics relationally within the time available. Yet taking into consideration the time necessary to a practical approach to teaching with large classes, a more appropriate solution would be to offer the MTCC over two semesters with a total of 160 hours. Increasing teaching time and reducing class sizes involve institutional changes. I have been trying to make these changes, but until the time of completion of this paper these problems have not been solved.

Discussion

The decision to ask STs to experience children's activities much more often than I had done in previous courses was thought to be appropriate as it did not cause any motivation problems in adult learners. On the contrary, the majority of STs said that they had enjoyed using children's activities. The games and practical activities with concrete materials and measurement instruments were time consuming and a hard work with large classes, but using children's activities proved to be an appropriate strategy to improve STs' relational understanding of mathematics since the majority of STs said, and many indicated in the post-tests, that their understanding had improved. The activities in the program did not require any changes in nature, mainly quantitative and timing adjustments were made for the second and subsequent semesters in order to maximise STs' learning during a single semester. More practical and written activities were gradually included for the representations and content that proved to be more difficult for STs in previous semesters. For this reason certain activities had to be excluded from the program.

Teachers' ability to translate SMK into mathematical representations is considered an important part of teachers' PCK (e.g., Wilson et al., 1987 and Ball, 1990). So it was necessary to help STs draw out clear connections between the symbolic ways of representing mathematics they had in their minds before starting the course and other ways of representing the subject so that different representations for the same concept or operation could be incorporated in the same schema. Providing STs with children' activities involving translations among and within multiple modes of representation was thought to help them (a) improve their own relational understanding of mathematics and (b) learn an important form of PCK in a tacit way. Acquiring a repertoire of representations and activities that can be transformed by the teacher for classroom use seems to be an adequate and initial form of PCK for a course component about mathematics teaching in pre-service teacher education. With time and teaching experience STs would be more able to use such knowledge in combination with more sophisticated teaching strategies.

I could have focussed my teaching on teacher development by adapting content, assessment, principles and aims. Narrowing or avoiding the teaching of contents which proved to be more difficult for my STs could make my classroom life easier. However, such actions were thought to be socially irresponsible because they would affect STs' learning of SMK and PCK and this, in turn, could limit their future students' mathematical learning. So I decided to focus the teaching on my social responsibility to primary school students. According to Darling-Hammond (1996), poorly prepared teachers are "assigned disproportionately to schools and classrooms serving the most educationally vulnerable children" (p. 6). Research results about effective schools also suggest that children from more privileged classes can compensate more easily for an inefficient teaching while children from less privileged classes are twice harmed if teaching is not effective (Mello, 1998). School failure and evasion are two issues commonly discussed in departments of education in Brazil. Yet it does not make sense to discuss those issues without providing STs with enough opportunities to acquire the SMK and PCK which could help them to deal more effectively with underachievement. An ideology of struggle and effort (Delpit, 1995) provides some support for my determination not to set low standards where an ideology of consumerism, students as clients, may have led to different decisions. The key question is "who is the consumer, the STs or their future school students?".

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Summary of the activities most often performed by the STs

The main children' activities are:

(1) Using diverse ways of representing mathematical concepts and operations

Activities involving translations among and within multiple modes of representation (contexts, concrete materials, pictures and diagrams, spoken languages and written symbols) for most concepts and operations in the primary school mathematics curriculum.

(2) Using diverse ways of performing operations

(a) Practical work and discussion about different algorithms for operations with natural numbers (with hundreds, tens and units) in the concrete mode (no symbols are used).

(b) Practical work and discussion using concrete materials and symbols to present and consolidate the traditional algorithms for addition, subtraction, multiplication and division with natural numbers.

(3) Focusing on mathematical relationships

(a) Activities involving translation between "versatile" representations for each concept with rational numbers. These are representations that can be used for two or more related concepts thus allowing their relationship to become more explicit. Some of the representations that are used for natural numbers (e.g., straws, part-whole diagrams, number lines and symbols) are now extended to fractions, mixed numbers and decimals in order to highlight the existing relationships between natural numbers and rational numbers.

(b) Practical work and discussion using "versatile" concrete materials and symbols to perform algorithms for addition, subtraction, multiplication and division with rational numbers (fractions, mixed numbers and decimals) that are extensions of the traditional algorithms for natural numbers. For example, in all five types of representation the algorithm for division of two natural numbers (e.g., 24 ÷ 3) is extended to the algorithm for division of a mixed number by a natural number (e.g., 24¾ ÷ 3).

(4) Paying attention to the transition from concrete to symbolic

(a) Formalisation activities. Through systematic questions asked by me, the STs are asked to look back at their previous actions with concrete materials and symbols (activities 2c and 3b) and verbalise their past actions (e.g., What did you do next with the tens blocks?). The objective is to construct a symbolic algorithm separated from the concrete materials. Each step in the symbolic algorithm is written by me on the chalkboard after each question is answered.

(b) Written exercises involving translations from pictures and diagrams to symbols concerning numbers and operations and the inverse translations (from symbols to pictures and diagrams). The STs are asked to do the last three items of selected children's exercises.

Other children' activities are performed by STs, but in a less frequent base like: (a) reading selected paragraphs about the history of number systems, fractions and decimals extracted from children's books; (b) playing games in pairs, (c) counting forwards and backwards with mixed numbers and decimals, and (d) performing practical work and discussion using concrete materials and symbols (after children's activity 2a), with the aim of comparing two specific concrete algorithms for addition and division of natural numbers and decide which was the quickest way of finding the solution and why: (i) starting from the hundreds (left to right), or (ii) starting from the units (right to left).

The main teachers' activities are:

(a) Listening to exposition about some ideas concerning the use of mathematical representations from the theories of Jerome Bruner, Richard Skemp and Zoltan Dienes. I present these ideas in the first week of the semester and revise them after some related activities by asking questions such as: "Why did I use bundled straws in previous activities and now I changed to Dienes' base 10 blocks? What is the theoretical principle behind this change?"

(b) Participating in short methodological discussions about some of the activities and representations. I start the discussion with questions such as: "How can this activity be adapted to younger children?", "Which relationships can this activity help children construct?" and "Which is the clearest representation for place value: bundled straws or the abacus? ... Why?"

(c) Reading and discussing a few selected paragraphs extracted from the literature about teaching and learning specific mathematical contents.

(d) Recording the actions behind the algorithms performed with concrete materials and symbols (after children's activity 4a).

(e) Identifying misconceptions in children's work concerning operations with natural numbers.

(f) Answering tests with open-ended questions in such a way that relational understanding can be probed through a context of teaching children. For example: "How would you help your students to understand the reason for the result of ¾ x ½?" These tests were used as data during the action steps of the research.