REFLECTIONS ON THE TEACHING OF DIFFERENTIAL EQUATIONS: WHAT EFFECTS OF A TEACHING TO ALGEBRAIC DOMINANCE?

Salahattin ARSLAN*, Hamid CHAACHOUA* and Colette LABORDE**

*Leibniz, MTAH, and **IAM, Grenoble, France.

The notion of differential equation is introduced at the end of the secondary education, and then it constitutes a considerable field of study at the university. For several years the dominance of the algebraic in the teaching differential equations has been picked out in several researches (Artigue, 1989, Boyce 1994, Habre 2000, etc.). To contribute to these researches, we are interested, in this article, in the character object (According to (Douady, 1986)) of the differential equation and we propose to release some conceptions, stemming from an education with algebraic dominance, of the students of Bac+3 level in France.

INTRODUCTION

After having presented the object differential equation in a mathematical point of view in the first paragraph; we give, in the paragraph 2, some information about the teaching of this notion. Then paragraph 3 presents the questions and the methodology of the research. In paragraphs 4 and 5, we present our experiment and results stemming from it. Finally to finish, in the paragraph 6, we present the conclusions to which this study can lead to allow a better teaching of the object in question.

1. THE OBJECT DIFFERENTIAL EQUATION

The notion of "*differential equation*" is in rough terms defined as a “relation, valid on an interval, between a variable and its successive derivatives.” Writing of such equations represents an essential step of a process of modelling, nevertheless, so that this latter can be achieved, it is necessary to approach an another step which consists in resolving the differential equation. The task "*to solve a differential equation of the first order*" represents one of the problems of analysis most often encountered by the scientists, the engineers, but also by the students. The scientists formerly looked for methods to resolve such equations and to express their solutions under a suitable shape. Thus the scientific field saw first of all the emergence of two methods to determine the functions solutions:

The first one, called algebraic, allows to express the functions solutions in exact shape, that is in the shape $y=f(x)$. The obtained functions satisfy, as its derivative, the differential equation. In other words, it is a question of finding the primitive which generated the differential equation to solve. The second method, called numerical, consists in calculating the values taken by functions solutions in a few number of points. There are varieties of methods (e.g. method of Euler or Runge-Kutta), known as "*numerical methods*", to calculate this set of values, called "*numerical solution*". Let’s note that this resolution mode doesn’t provide an approximation of the curve solution with every point but rather in points of given abscissas, then the estimate of the solution in other points can be obtained by interpolation.

It is only at the end of the 19th century that Poincaré contributed to this field by the inauguration of what is called "**the qualitative theory/approach**" of differential equations. As its name infers, this <u>third method</u> consists of a qualitative study of a differential equation of the shape $y' = f(x, y)$, without solving it. In fact the observation of this last one allows to capture vital information concerning curves solutions.

A finer study of these three modes of resolution allows to say that each of them has its advantages and its disadvantages. For example the algebraic resolution is an algorithmic technique which satisfies thus teaching expectations. However application of such a technique is not always possible and depends strongly on the nature of the differential equation (linear or not; with or without second member etc.). Even for a differential equation of the shape $y'=f(x, y)$, when it is nonlinear; there is no general shape to represent it and no categorical information can be given. So it is impossible to propose suitable general methods to approach them. It follows that the recourse to the other methods of resolution is necessary. Moreover we postulate that *the qualitative approach improves students' knowledge of certain key notions of the analysis and thus plays an indispensable role for the idealisation of their knowledge*.

2. THE CURRENT TEACHING OF THE DIFFERENTIAL EQUATIONS

The analysis of programs and textbooks shows that the differential equations are studied with an algebraic approach in the last year of secondary school where they are introduced into the educational environment in France. This conforms to the tendency of the teaching of the analysis where the algebraic techniques are privileged because they offer algorithms of resolutions which are synonymous with a certain assurance for the pupils, but also for the teachers.

For the university teaching, it is difficult to determine the curriculum because the teaching's contents are not always the same between universities and even from a teacher to the other one. Thus, for the differential equations course a few teachers have recourse to the other approaches, numerical or qualitative, and there is not always an articulation between those approaches. Nevertheless, we can notice that the algebraic approach remains widely dominant in the university level like other studies affirmed.

We should never wrongly believe that such a reduction to the algebraic resolution is a phenomenon specific to France. Quite the contrary it is a phenomenon which exceeds the French borders. As Habre (2000) report, it's the same for example, in the United States: the latter established that most of the students were tempted by the algebraic resolution and that little among them was showed enticed by the qualitative approach and finally that most represented a reserve to this approach.

3. THEORICAL FRAMEWORK

In the previous paragraph, we have underlined the dominance of the algebraic techniques for the teaching of the differential equations. However several factors reveal shortcomings of these techniques: apart from the existence of the differential equations that we cannot algebraically solve, even in case we can express all the solutions under formula, the calculations necessary for the resolution are generally voluminous and boring. In addition there are singular solutions, not engendered by the general solution.

Naturally these constraints constitute obstacles for the enrichment of the types of the differential equations in the education which sees itself more secure with limited types of skilfully chosen equations. In spite of the demand of the programs, textbooks remain poor concerning a teaching around the activities of resolution of problems rich, significant and more meaningful. Such a selection is not naturally without loss. It indeed brings to neglect the nonlinear differential equations which are more meaningful and give stability to the education. As a result of this choice, firstly students have difficulty in recognizing a differential equation, especially nonlinear ones and secondly we are faced, notably among the beginners, with an association "differential equation <-> exponential." In addition, Artigue (1989) indicates that such a situation evokes at the student's the illusion "*any differential equation can be integrate*."

In addition, the current education favours neither change of setting[1] nor the change of registers of semiotic representation[2]. However, we come round to Douady on the importance of the change of settings to endow a mathematical concept with a more meaningful character: the change of settings is a means to obtain different formulations of a problem which allows both a new access to the encountered difficulties and the implementation of tools and techniques which were not tangibles in the first formulation. An interplay setting thus gives to the reader objects and techniques which don’t exist in the setting where he works initially. So it allows to solve a problem difficult to solve even insoluble within the setting of origin, *by benefiting from differences between working domains*. (Douady, 1986 and 1992).

With regard to activity of the *coordination of registers*, for Duval (1995), it is indispensable to the mathematical activities. The latter gives three fundamental reasons to justify the vitality of the coordination of registers: It allows an **economy of treatment**: each register having its specificities; it may be that the treatment of a problem in a register is faster than in another. The plurality of registers thus gives the possibility of choosing the optimal one which allows

[1] A setting is defined as a set of the objects of a branch of mathematics, relations between these objects, and mental images associated with these objects and relations (Douady, 1986).
[2] Duval defines a "register" as a semiotic system satisfying four conditions: to consist of identifiable traces like a representation of something; to have rules of transformation to produce other representations; to have rules of conversion towards another system of representation; to have rules of conformity for the constitution of the units of higher level.

making treatments in a most economic possible way. The second reason is the **complementarity of registers**: it may be that a register represents only a part of the content to be represented. Hence to represent entire content, it is necessary to combine the different registers of representation of this same content. Finally a last reason is about **construction of knowledge**: the mathematical activity implies the coordination of the registers of representation. A subject (i.e. student) must to reach the level of the conceptual coordination of heterogeneous semiotics representations, so that he may able to discriminate between the representative and represented, or between the representation and the abstract contents that this representation expresses, instances or illustrates. (Duval, 1995)

It indicates that the coordination of several registers of representation is fundamental for the conceptual apprehension of mathematical objects. Later researches indeed showed that study in single setting or register is not favourable to the learning. These researches also emphasized the undesirable effects and the cognitive limits of the restriction on only one **setting / register**. It appears that the phenomena of changes of settings and registers help considerably to the teaching and the learning of an object. In our case, going beyond of the resolution algebraic by integrating both others methods, numerical and qualitative, implies other settings (in addition to algebraic, the numerical and geometrical one) and other registers (graph, numerical, tables etc).

In addition, we consider that the three approaches contribute to enrich the meaning of a differential equation and that they must be considered as complementary approaches. Besides the limitation on the algebraic approach deprives of the fertility of the changes of settings and of registers, from which our research hypothesis:

Hypothesis: the limitation to the only algebraic frame for the treatment of the differential equations can be at the origin of the misconceptions at the pupils.

4. RESEARCH METHODOLOGY AND EXPERIMENT

To test this hypothesis; we observed a module on the differential equations set up by a teacher within the framework of the training of the trainees of the IUFM[3] (Arslan & Laborde, 2003). The observation took place at three groups who had 3 years a scientific university formation with a speciality in mathematics and who prepare for the competitive examination of the teachers. The principal aim of this module which has existed for a few years consists in breaking, being based in particular on work of Mr. Artigue (1989), with the algebraic approach. Taking into account the agreement of our assumptions with those of the trainer (the contribution of the software to the teaching, the dominance of algebraic and the wish to attenuate it, the effect of the interplay settings and registers in the achievement of such objectives and the integration of the numerical and qualitative approaches etc.); it appeared

[3] **I**nstitut **U**niversitaire de **F**ormation des **M**aîtres (University Institute for Teachers' Training).

sufficient to us to take part in this organisation as an observer. We were thus satisfied to contribute to this module by proposing light modifications.

This observation intends firstly to identify the conceptions of the students on the differential equations and undesirable effects of algebraic dominance and secondly to measure at the same time the impact of the integration of other approaches: cost, viability, accord with students' knowledge; and thirdly to identify the contribution of a geometry dynamic software. For this didactical engineering, the students are thus use Cabri (Bellemain F., Capponi B. (1992) and Bellemain F., Laborde J. M. (1994)) which must favour the appearance of new implicit reasoning at the students. We think indeed that the field of the differential equations is a field for which one can expect that computer brings a dynamic help to the teaching, in particular by authorizing not easily realizable interactions in the environment paper/pencil, by favouring interplays settings and registers and by facilitating conjecturing. During the experimentation, composed of two sessions, four worksheets were proposed to the students:

The **first worksheet** aimed to explore the preliminary knowledge of the students on the object differential equation. Two activities were proposed: a first demand, for the differential equation $y'=2y$, to draw the tangent line in the point M to the curve solution at this point which is supplied as well as an orthogonal plan. The coordinates of M are not explicitly supplied, but they can be very easily identified thanks to the drawing. With regard to the second activity, we proposed them a series of expressions and asked to specify, for each of them, if it represents a differential equation and in that case, to indicate the dependent and independent variables. The activity proposes a range of differential equations the widest possible: of the first to the second order, with constant and variable coefficients, with and without second member, linear and nonlinear. With regard to the **second worksheet**, it has been passed following a complementary work for initiation to Cabri and concern exclusively the differential equation $y' = y$. It is a question, in this worksheet consisting of eight questions, to combining the three modes of resolution of the differential equations: the entrance is made by a graphic study, and then the algebraic resolution is added to this panoply and finally the numerical resolution. Cabri intervenes in any level: to construct the tangent vector, to plot a curve solution verifying given initial conditions or still to use the tangent vector, drawn thanks to the software, to construct an approximate curve of the equation via Euler's method etc. The **third worksheet**, composed of three activities, can be divided into two parts: the first two activities which concern the concepts of "curve and function" and the third activity where we proposed to the students to associate, with a justification of their choices, variable tangent vectors pre-drawn in Cabri with one of the differential equations provided. We have favoured in this task the formulations of the conjectures via software about properties of the differential equations in graphic setting having consequences in the algebraic one. The conjectures must lead to the choice of the suitable equation.

However at the end of the experiment with the first group of trainees, we noted that the students implemented undesirable rules such "to calculate the slope at given point and to compare it with the figure" or "to try to guess, via “Trace” tool of Cabri, the shapes of the curves solutions to compare them with the general solution of the differential equation solved algebraically." However we expected in fact they detect geometrical invariants to connect them to suitable differential equation. For example when the vector is vertically unchanged, it must be associated with an equation depending only on variable *x*. In the contrary case, it is associated with an equation depending on two variables. To block such strategies, we proceeded to a modification in the differential equations and we replaced the coefficients by parameters. For example, the equation $y'=y^2/x^2$ then became $y'=by^2/x^2$ ($b\in\Re$). Lastly the **fourth worksheet** consisting of five activities, is about qualitative study of $y' =y^2-x$.

5. RESULTS OF THE EXPERIMENT

Among the present students, a few have already used Cabri and a few already claimed to have made a qualitative study but nobody heard about the concept of isocline for example. This shows once again the dominance of the algebraic. In addition the questionnaire enabled us to measure the impact of the university education, dominated by algebraic, on the conceptions of the students relative to the differential equations. We thus noted that the dominance of the algebraic is an obstacle for a qualitative study. From this point of view, analysis of the difficulties will be based on the dominance of the algebraic. In this article, we will not present whole results of this experiment which analysis is in progress but we will be satisfied to communicate some significant results, resulting from the two activities of the first worksheet and the second activity of the third worksheet, presented below.

Concerning the activity of the **draw of the tangent line (Worksheet 1, Question 1)** the students can choose among the two following procedures to achieve the task:

P1: to determine the coordinates of M and to calculate then the slope of the tangent line via differential equation;

P2: to solve the differential equation to obtain the general solution from which they determine the equation of the curve solution at the point M. This equation is then derived to calculate the slope of the tangent line. In addition, it may be that the students, without solving the equation, refer to the shape of the curve solution which they know inside out hence they draw the tangent without any justification.

The analysis of the productions of the students confirms the tendency of the students towards the procedure P2 (18 out of 32) and there were only three students having recourse to P1. Four students, on the whole, correctly realized the plot, five students didn’t treat the activity, two students gave incomprehensible answers and finally four students confused differential equation with the equation of the tangent line. For example one affirms that "the tangent line at M is the straight line with equation $y'=2y$." In regard to students resorting to P1, all the three succeed and we can say that they have a good mathematical interpretation of differential equation. Among the students resorting to P2 only one succeeded. It follows that the students were incapable to use the general solution to succeed, however they could have made it as mentioned above. The existence of P2 can thus be interpreted by an automatism at the students who just satisfy the didactical contract, indeed a differential equation evokes at these students the "algebraic resolution": "when we are faced with a differential equation, it should be solved, it is used for nothing else, at least in the course of mathematics." The fact that they don't succeed to trace the tangent line or to calculate the slope might have its origin in the following conception, inherited from an algebraic teaching: "*to study a differential equation, we need absolutely an algebraic expression. Event it is a question to calculate a slope at a point*."

Let's note in addition that, among these students having recourse to P2, four didn't even succeed to solve the equation. It is the case of three students who gave the general solution of the form $e^{2x}+K$ instead of Ke^{2x} ($K\in\Re$). In addition, a student confused the differential equation provided, with $y'=2x$, hence the resolution is made in the following way: « $y'=2y \Rightarrow \int y' = 2\int y \Rightarrow y=2y^2+K$. » ($K\in\Re$). This confusion can in fact be connected to the following conception: "in a differential equation of the form y'=ay ($a\in\Re$), the unknown "*y*" represents at the same time an independent and a dependent variable." Concerning the identification of the solution at the point M from the general solution, the situation is worst: eleven students (out of 18) treated it in an erroneous way.

We thus observe the difficulties of the students when they are faced with qualitative interpretation tasks which break with the didactical contract. Apart from three, the students didn't know provide a mathematical interpretation of differential equation –*y'* is not considered as a function- and hence in general they didn't recognize that one can calculate the slope by means of differential equation. The behaviours of the four students, who identify differential equation with the equation of the tangent, show the difficulties in geometrical interpretation of the differential equation. Besides following activity highlights how the concept of derivative is conceived. For example a student proposes following factorization: $ax'^2+bx'+c=a(x'-x1')(x'-x2')$.

In addition, the majority of the students have difficulties in giving the names of the dependent and independent variables but also in **identifying a differential equation (Worksheet 1, Question 2)**. For this question, we won't present quantitative information but we will just quote some striking phenomena. For example for certain students a differential equation can be defined without "**any**" variable when another reduces the differential equations to expressions of the form $x'=a$ or $y'=b$ ($a, b\in\Re$). In addition, for one student the only criterion to identify a differential equation is its solubility. Thus he refuses $ax'+by'=0$ - for which another student proposes *a* and *b* for the independent variables – because it "*is not soluble (in dimension 4)*." He proposes as variables *x* and $y=x+b$ ($b\in\Re$) for the equation $y'=1$, whereas for $2y''+3x=0$ he proposes *x* and "*polynomial of the type* ax^3+bx+c" which is undoubtedly considered as the general solution of corresponding differential equation.

Following the example of the student who gives as reason the existence of "*two different variables* " to reject $y''+3x=0$; we also noted that an equation with second member can pose problems for the students. The nonlinear equations pose also difficulties for them: for example, a student introduces "*y polynomial*" with " $y= gac+d$ $(d°\leq 1)$ *(if a and* $c\geq 0$*)* $g, d\in \Re$ " for the function and *x* for the variable for the nonlinear equation ($ax'^{2}+bx'+c=0$) which is not admit as a differential equation by 12 out of 32. For example, a student accounts for his response by saying that "*one seeks the solution y for the equation* $ay^2+by+c=0$ *and* $y=x'$ *and one finds x*." Then the function is associated to the "exponential", whereas the independent variable is provided like $x'=x_1'$ or $x'=x_2'$. This shows existence of the following conception: "*the general solution of any differential equation inevitably possesses the exponential function*."

This analysis confirms firstly the deficiency of the algebraic education, incompetent to provide to students tools to recognize a differential equation, in particular nonlinear ones which don't represent in general a privileged field for the teaching and secondly it explains difficulties of the students in giving meaning to a differential equation.

Finally the data collected for the activity where it was a matter of **connect variable tangent vectors to one of the provided differential equations (Fiche 3, question 3)** reveals that in spite of our demand, the students didn't seek to argue the behaviour of the tangent vector by connecting it to functional expression of *y'* and that the first strategies of resolution consist in identifying the good equation by trying to guess algebraic expression of the curves solutions or to calculate the slopes at the points randomly chosen and then to compare these numerical values with the slopes of the vectors on the screen of the software. In new study where we tried to block the recourse to these algebraic processes, the students had to seek other process of resolution. Some were then brought to hold reasoning connecting the graphic behaviour of the tangent vector to the characteristics of the expression of *y'*. Hence we could note that in such study emerged spontaneously some new reasoning connecting graphic and algebraic at the students to associate the tangent vectors with differential equations. We also noted that when the algebraic resolution was accessible, the recourse to these procedures was more important than in the case the algebraic resolution wasn't accessible to the students. It seems that this is a didactical variable at teachers' disposal to favour a qualitative approach.

This analysis allowed us to note that the possession of the algebraic techniques set up an obstacle to the learning of the qualitative approach. We also note the same tendency at the Mexican students: Moreno & Laborde (2003) gave students of the second year of the Mexican universities activities where having a removable curve in the screen of Cabri, they had to associate it, "WITHOUT INTEGRATING" with one of the differential equations provided in a list. As in our case, expectations of the authors consist in the fact that the students rely on "geometrical invariants" to succeed. They reported on this matter that the strategy of solution consisted generally in trying to integrate the equations and to obtain then the curves solutions. They also collected remarks of the form "*one cannot make the choice without integrating, so one will integrate them mentally*."

In addition Rasmussen for his part reports that given a field of tangents, when he requests to the students to trace an approximate solution at a given point by means of this field of tangents, these students produce layouts which resemble their exact solution and with nothing to do with the field of tangents.

All this highlights how the algebraic resolution obstructs the integration of the geometrical approach, especially at the beginner pupils. In addition restricted to the algebraic resolution, students do not give any meaning to the differential equations.

6. CONCLUSIONS

In spite of the algorithmic character of algebraic approach, it involves several techniques (integration by parts, partial fraction decomposition etc) which are difficult to retain continually. Hence students risk forgetting all methods of resolution which they learnt. It may be the reason for which some students of IUFM were unable to solve differential equation $y'=2y$ and other proposed "ax^3+bx+c" as general solution for $2y''+3x=0$.

In addition, we noted that the students were also unable to propose a geometrical interpretation because the majority of them had difficulties in giving the names of the dependent and independent variables and in identifying such an equation.

This analysis highlights that the restriction of the teaching of the differential equations on an algebraic teaching emerges the consequences rather undesirable. Although one believes that those are attenuated as and when the students advance in the school level, unfortunately this is not the case. Indeed, the experimentation carried out with the future teachers revealed some misconceptions and difficulties, dependent on the restriction to the algebraic setting. Such reports result in blaming the naive belief on progressive attenuation of these dangers and lead us to join us in Artigue (1996) who highlights the existence and the resistance of the difficulties in the field of the analysis which she prefers to group together into three, of which are "*difficulties linked to necessary breach with the algebraic thinking*." To enter an analytical thought, it is thus absolutely to break with the algebraic thought. In addition, in consideration of the productivity of the interplays of settings and registers, we believe in the necessity of integration of the other approaches (qualitative and numerical) for the differential equations' teaching. Thanks to such integration, we think that it would be possible to face the difficulties, and erroneous conceptions developed in the preceding paragraph.

Finally we think that software can bring a considerable help to teaching by making easier the passage among settings/registers. The analysis shows that software provides tools which one doesn't lay out in a paper/pencil environment. Thus it allows the conception of new tasks favouring the interplays settings and registers, and also allows an emergence of new reasoning at the students.

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