A COMPUTERIZED INTERACTIVE APPROACH TO REAL NUMBERS AND DECIMAL EXPANSIONS.

Laure Barthel

Hadassah Academic College, Jerusalem, Israel.

Students who are entering College or University have idiosyncratic notions of real numbers and of their decimal expansion in particular. These "concept images" are the source of conflicts and confusion when the students encounter mathematical material, such as calculus, where the specific properties of the real numbers matter. In addition high school teachers need to be better aware of the nature of the real numbers relative to other number systems. We present here the project of a computerized interactive module which familiarizes the students in a concrete way with the notion of infinite decimal expansions of real numbers.

INTRODUCTION

Real numbers are probably the most widely used mathematical object during high school studies, but they are never properly introduced at this level. By the time the students enter more advanced studies, they have built for themselves an image of real numbers based solely on their experience. Using the terminology of Tall and Vinner (1981), this creates a huge gap between their concept image of the real numbers (entirely of the type of *informal imagery* (Vinner, 1995)) and the concept definition which does not exist at that stage.

If their University/College studies include mathematics in its curriculum, one of the first courses that they will encounter is usually calculus, in various degrees of formality. As Spivak (1994) dramatically demonstrates in his Calculus' textbook, the completeness property of the real numbers is the basis of the most fundamental theorems on continuous functions: the mean value theorem, and the two Weierstrass theorems. This property takes us away from the purely algebraic nature of previously encountered numbers, such as integers or fractions, and introduces objects of infinite nature (expansions, limits...). If one wants to study calculus in a meaningful way, one cannot continue to take the real numbers for granted.

In fact the issue is not a trivial one and one should remember that first year students are not the only ones who have been puzzled by real numbers. From Zeno's paradox, through Leibnitz' infinitesimals, the history of mathematics presents a long list of endeavors to grasp the notion of real numbers. They have culminated with the formal classical set-theoretical definitions (the completion of the rationals via equivalence classes of Cauchy sequences or via the Dedekind cuts). The definitions are usually used at the beginning of the real analysis course for mathematics' students. This formal construction is considered advanced material and is very difficult to grasp even for beginning students. For that reason, this part of the syllabus is usually skipped over in calculus courses that are not intended for mathematics majors.

The lack of comprehension of the nature of the real numbers has wider consequences than the understanding of a specific course. Mathematics teachers in training for example should know them at a much deeper level than the sum of their prior acquaintance. In another field, computer scientists need a good understanding of the difference between a real number and its decimal approximations in order to appreciate the difference between a real number and its representation for a computer, and also to address the issue of robustness of an algorithm correctly.

For all these reasons, T. Leviatan (2004) had started a project of a transitional course centered around number systems, where the real numbers wouldn't be introduced through an abstract set-theoretic definition but rather through the concrete notion of decimal expansion. Their properties are investigated in a rigorous and explicit manner. A similar approach has been described, for example, by Tim Gowers (web pre-print). There are several reasons to insist on defining real numbers through decimal expansions. Not only are they more concrete than the alternative set-theoretic definition: in fact, decimal representations are what students have previously seen and will continue to see when discussing real numbers. For this reason it is necessary to reconcile the students' concept image of real numbers with decimal expansions.

In this article we aim to explain how such an approach can be supplemented with great benefit by a separate software unit that we are developing in parallel. Our software module consists of interactive Mathematica notebooks illustrating various aspects of decimal expansions. Although this project has originally been developed for training teachers, it is suitable to other branches of higher education. In particular, the software module would give beginning students in other fields the opportunity to deepen their understanding of decimal expansions and real numbers.

In the next section we recall some well known misconceptions that students have regarding real numbers, decimal expansions, and rational numbers. We also give certain goals that we hope to achieve in increasing the understanding and fluency of manipulation of these notions by the students. We then explain how the use of symbolic mathematics software such as Mathematica helps one to achieve these goals. In the last part we shall describe specific examples of interactive work are expected to do.

MISCONCEPTIONS REGARDING REAL NUMBERS AND DECIMAL EXPANSIONS.

The students' misconceptions about real numbers have been described in many studies. We will recall a few of them that we particularly wish to address in our project.

A real number is often considered as equal to its partial decimal expansion (typically, the decimal expansion given by a standard calculator). This is in fact a natural attitude to develop after years of using real numbers in an applied way. As is remarked in Tall and Schwarzenberger (1978):

> We only actually need a very few places for any practical degree of accuracy. The well known approximations $\sqrt{2} = 1.4142$, $\pi = 3.1416$, $1/3 = 0.333$ illustrate the fact that we do not bother to quote more than 4 places of decimals, and even this accuracy is far beyond what can be usefully employed in a practical application.

For example, first year students in computer science were asked in a discrete math problem to prove by induction the well known formula for the n'th Fibonnacci number:

$$F_n = \left[\left(\left(1+\sqrt{5}\right)/2\right)^{n+1} - \left(\left(1-\sqrt{5}\right)/2\right)^{n+1}\right]/\sqrt{5}\,.$$

Most of the students "proved" the basis of the induction by checking the cases $n = 1,2$ using their calculator. For them, the number 2.2360680 that appears on the screen of the calculator when they type sqrt[5] *is equal* to $\sqrt{5}$. As we mentioned briefly in the introduction, the difference between exact computations with real numbers and computations with partial decimal expansions becomes later crucial the notion of robustness in computer algorithms. Let us consider a concrete example for more clarity: in the course of many geometrical algorithms it is necessary to test whether, given three points p, q, r in the plane, the point r lies on the line from the point p to the point q or if not, on which side of the line it is. This is easily done by computing a 3 by 3 determinant. But if the points are nearly collinear, the computation can give an erroneous result, causing the whole algorithm to fail.

It is clearly of importance to reconcile the notion of decimal expansions with fractions. In particular, the decimal expansions of a fraction are sometimes, but not always, finite. In fact, one of the standard results about decimal expansions that most students have learned at some point is the periodicity of the expansion for rational numbers. It is usually presented as a nice consequence of the long division algorithm. But as Pinto and Tall (1996) find in their study, students tend to have "*idiosyncratic evoked concept images of rationals*". Expansion like $0.33333...$ or 0.25 are readily recognized as rationals, but

> The finite decimal 0.97853 and the recurring decimal 0.343232… were then classified as irrationals because in each case she could not imagine a fraction that could represent it.

The wording of the student's quotation shows the source of the misconception: "*she could not imagine"*. This student, like the others, had seen very few examples of decimal expansions of rational numbers. As Pinto and Tall write about another student:

> She appeared to be working with a mental list of specific examples of irrationals (such as π and $\sqrt{2}$) and rationals included numbers such as $3/5$ explicitly written in fraction form, but if the expression could not be readily converted into rational form *by her*, it was considered irrational.

This lack of examples forming the concept image is often a limiting factor in the mathematical performance of students. This has been illustrated by Alcock and

Simpson (2002) in the case of the understanding of convergent sequences. Regarding decimal expansions, this is even accentuated by the fact that the students have seen many truncated decimal expansion when the computed the numerical value of the result of a geometric problem (often an irrational number, with π for example) or the result of an elementary probability problem (often a fraction). The irrational and the rational answers they got from their calculator look similar because there are very few fractions where the periodicity of the decimal expansion is visible on a calculator.

Lastly, a famous example of misconception is that most students think that 0.9999… is smaller than 1 (see, for example, Tall and Schwarzenberger, 1978; or Tall and Vinner, 1981). Some reasons for this confusion are "*the lack of understanding of the limit process …the misinterpretation of the symbol* $0.\dot{9}$ *as a large but finite number of 9s"* (Tall and Vinner, 1981). In fact, this difficulty in grasping the notion of limit and the tendency of identifying a real number with its truncated decimal expansions, as we mentioned at the beginning of this section, are probably related. On the one hand, there are the real numbers as the students think of them, a decimal with a few digits in its expansion. On the other hand, there is the abstract notion of limit. It has or has not been formally defined, depending of the case, but most students have at least encountered it in the context of the sum of geometric series. But the students' intuition separates these two sides. There is no "feeling" of the real number as the limit itself.

COULD COMPUTERS HELP?

All these examples have led us to the idea of using software like Mathematica to illustrate the construction of real numbers by their decimal expansion. The main advantage of such software is the possibility of doing computations to any amount of precision. As we have seen above, the imagery of decimal expansions of the students is very limited, mainly because it has been built from their experience. A pocket calculator is limited to a fixed number of decimals, and calculation done by hand is in practice limited to even fewer decimals. Mathematica (and other similar software) features give much wider possibilities:

- It is possible to see 50, 100 decimals, which gives a much better intuition of the infinitude of the expansion.
- The length of the expansion can be dynamic, and not fixed as on the calculator. One can compute many decimals, and then, to get a better precision, it is possible to compute even more.
- Thanks to the computing power of the software, examples are not limited to simple cases.
- It is easy to build a program so that students can easily change the initial data, making the computerized unit interactive. This gives to the students the opportunity to experiment with many more examples that can be covered in a

lecture and to take an active part in the learning process. As Malabar and Pounty (2002) write:

> Knowledge, as discussed in the previous section, is built up from personal experiences, and making these experiences more dynamic will assist in the development of cognitive structures.

Moreover, the possibility of extending the decimal expansion as much as needed and of showing the increasing degree of accuracy of the expansion gives a constructive introduction to the notion of limit. In the more advanced parts of the computerized module it is even possible to present the problem of defining the sum of two real numbers from their infinite decimal expansions, which relates to the issue of the arithmetic of limits.

As experience grows it is possible to add parts to the interactive module addressing specific misconceptions or emphasizing specific needs.

In summary, our goal is to use the computer to enlarge the concept image that students have of the real numbers and to give a concrete introduction to several issues concerning the concept of limits.

PRESENTATION OF THE COMPUTERIZED MODULE.

We shall now give a few examples of activities about real numbers planned in the project.

Construction of some decimal expansions.

This unit illustrates the constructive definition of decimal expansions for numbers like square roots.

Decimal expansions of rational numbers.

This unit deals with the correspondence between rational numbers in their fraction form and rational numbers in their recurring decimal expansion.

The input being a fraction, the unit will show the successive remainders under the long division, thus underlining the periodicity of the expansion. The final output is the periodic expansion. It is thus possible to see periodic expansions that are much less simple and much less obvious that the common examples. Examples like $1/49$ show a very long period and enlarges the common image of a periodic expansion that looks like $0.3333...$.

The input is the integer part, the non periodic part of the decimal expansion, and the period. The unit will reconstruct the rational number and the output is the rational number in a fractional form. It gives as an intermediate result the geometric series induced by the periodic part of the expansion. The goal is to help build the imagery of the students so that the rational form of a number given by a recurring decimal expansion becomes part of their mental image.

Arithmetic on decimal expansions.

We shall describe here the case of the sum of real numbers. Multiplication and other operations can be similarly presented.

The first step is a review about the sum of two numbers given by finite decimal expansions. The addition is done from right to left, once the decimal expansions are made to be of equal lengths.

The second step is to illustrate the problem of adding two infinite decimal expansions: in that case the normal addition from right to left is impossible. Recalling that two infinite expansions are the limits of their truncated forms, the unit computes the sum of the successive truncated expansion. One can see then how some digits can be changed by the carry from successive digits. We can experiment to find conditions where we can be sure that the digit will not change. Let us look at an example: suppose that up to 8 digits two numbers are given by the following expansions: $a(8)=0.23836363$ and $b(8)=0.66563636$ (the notation $x(k)$ means the finite decimal expansion of the number x with 8 decimals). Then we get the following truncated sums: $a(1)+b(1)=0.8$, $a(2)+b(2)=0.89$, $a(3)+b(3)=0.903$, $a(4)+b(4)=0.9039$, $a(5)+b(5)=0.90399$, $a(6)+b(6)=0.903999$, $a(7)+b(7)=0.9039999$, $a(8)+b(8)=0.90399999$. As it is easily shown, whenever we get in the sum a digit that is smaller than 9, we know that the preceding digits will no longer change. Thus in this example, the two first digits are fixed: $0.90\ldots$. But the following digits could change. There are three cases that can occur, according to the expansions $a(9)$ and $b(9)$. Here are some typical example for each case:

- $a(9)=0.238363632$ and $b(9)=0.665636363$. Then $a(9)+b(9)=0.903999995$ and the first 8 digits are now final.
- $a(9)=0.238363632$ and $b(9)=0.665636369$. Then $a(9)+b(9)=0.904000001$ and the first 8 digits are now final.
- $a(9)=0.238363636$ and $b(9)=0.665636363$. Then $a(9)+b(9)=0.903999999$ and still only the first 2 digits are final.

This example shows how it is possible to get a decimal expansion of the sum that is a good approximation of the sum, but nevertheless does not have the correct decimals.

A third step deals with the sum of rational numbers. On the one hand it is possible to add algebraically the sum of two rational numbers through their fractional form and get an exact result. On the other hand, it is possible to compute the sums of the successive truncated expansions and compare them with the finite decimal expansions of the exact result.

CONCLUSION

These few examples illustrate how we can use Mathematica notebooks to achieve our goal of expanding the imagery of real numbers of students and deepening their understanding of their decimal expansions.

REFERENCES

Alcock, L, & Simpson, A. (2002) Two components in Learning to Reason Using Definitions. *Proceedings of the Second International Conference on the Teaching of Mathematics, 1-6 July 2002, Hersonissos, Crete.* Retrieved April, 2004, from http://www.math.uoc.gr/~ictm2/Proceedings/pap110.pdf

Gowers, T, "What is wrong with thinking of real numbers as infinite decimal?" Retrieved April, 2004, from http://www.dpmms.cam.ac.uk/~wtg10/decimals.html

Leviatan, T., (2004). Introduction of Real Numbers, When and How? Preprint.

Pinto, M., & Tall, D.O. (1996). Student Teachers' Conceptions of the Rational Numbers. *Proceedings of PME 20, Valencia,* 4, 139-146.

Malabar, I., and Pountney, D.C. (2002). Using Technology to Integrate Constructivism and Visualisation in Mathematics Education. *Proceedings of the Second International Conference on the Teaching of Mathematics, 1-6 July 2002, Hersonissos, Crete.* Retrieved April, 2004, from http://www.math.uoc.gr/~ictm2/Proceedings/pap227.pdf

Schwarzenberger, R.L.E., & Tall, D.O. (1978). Conflicts in the learning of real numbers and limits, *Mathematics Teaching,* 82, 44-49.

Spivak, M. (1994). *Calculus.* Houston: Publish or Perish, Inc.

Tall, D.O., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity, *Educational Studies in Mathematics,* **12** 2, 151-169.

Vinner, S. (1991). The Role of Definitions in Teaching and Learning Mathematics. In D.Tall (Ed.) *Advanced Mathematical Thinking,* Kluwer: Dordrecht, 65-81.