THE TEACHING OF CALCULUS AT THE TRANSITION BETWEEN UPPER SECONDARY SCHOOL AND UNIVERSITY: FACTORS OF RUPTURE.

A study concerning the notion of limit.

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Abstract : The mathematical work in the field of Calculus is usually very difficult for even good students when they are entering the University; we have studied the transition between the secondary mathematical organisation in teaching (pre)calculus – actually the notion of limit, and the University one. Our study leads us to identify numbers of ruptures in the way students are required to work from upper secondary school to University; we identify didactical variables which values reveal a major change from one institution to another. We characterise the ruptures with the nature of the tasks, the role of the formalism, the flexibility, the use of different settings, the level of operation of the notion (process or object, able to be mobilised or not…). Finally, the utilisation that we make of semiotic representation settings proves to be a support to detect no exploited potentialities at the University. Our conclusion is that University does not take care of the previous knowledge students get when they enter this new institution; and we offer some hints to find a better mathematical organisation of this transition.

Noticing that mathematical work in the field of Calculus is usually very difficult for even good students when they are entering the University, we have studied the transition between the secondary mathematical organisation in teaching (pre)calculus, and the University one. We chose the concept of limit because it is the first analytic concept the students meet, and it is possible to build a very rich and contrasted corpus of tasks about limits, from the third year of scientific secondary school – that counts four years in Tunisia, but the study could have been similar in France from the Premiere and Terminal – to the first year of University.

We used various theoretical tools for that study, trying to seize the multiple dimensions of the work in either institution, and of the transition. Finally, we succeeded in characterising the ruptures with the nature of the tasks, the role of the formalism, the flexibility, the use of different settings, the level of operation of the notion (process or object, able to be mobilised or not…). This work led us to identify didactical variables – in the sense of Theory of Didactical Situations – that are pertinent to characterise the extent of the rupture. Analysing various exercises about limits, at the University as in secondary school, we found that our criteria were operational, and that a large variation of many variables could explain that even good students could not succeed in the work they are expected to do when they enter the University ; and, that the University did not take care of the nature of the knowledge students possess at the end of their secondary course. For example, University does not use the graphic setting anymore, as students are used to work with it and to understand properties with graphics.

In a first part we explain the way we used theories to analyse the transition ; and then we present our results about the main variations of didactical variables and their consequences.

I. Theoretical tools to identify didactical variables

I.1 THEORETICAL BACKGROUND

We started therefore by evoking tools introduced by Robert (1998) in this domain. These concerned the specificity of the notion of limit. Particularly, we putted out different functionalities of the limit: those that showed this concept as an unifying and simplifying one, and those that explained the work of formalism and generalisation at the beginning of the University.

Bloch (2000) made a step forward in clarifying the new requirements at the beginning of University by noticing that the entrance in a demonstrated Analysis demanded an understanding of the theoretical game rules, and especially of the validation elements of the theory (Analysis).

The anthropological theory of Chevallard permitted to analyse the mathematical activities after modelling them in "praxéologies mathématiques" (type of tasks, techniques, technologies, theories). And then, in reference to this theory, we studied the evolution of "praxéologies mathématiques" between the upper secondary school and the University, by focusing on the transient techniques and on the status of technologies (theorems, definitions, properties etc.) at each course.

In order to identify the level of operation of the notion which concerned the ability to be mobilised or not, we referred to Robert as she qualifies knowledge by technical, or summonable, or available.

However, according to Sfard (1991), who claimed that there is a deep ontological gap between operational and structural conceptions, we showed that the concept of limit can be defined both structurally and operationally. We tried to specify the work of conceptualisation in the considered levels, and to describe the flexibility to be built between the procedural dimension of this concept and the structural one.

Finally, the notion of semiotic representation's settings (Duval, 1993), informed us about the necessity of managing with several settings of representation (graphic, algebraic, analytic, etc.), and to be able to do autonomous conversions to pass one to the other.

In every case, this first phase enabled us to identify didactical variables that permitted us to detect important variations. The change of values of these variables are responsible for global ruptures between the secondary course and the University one.

I.2 DIDACTICAL VARIABLES TO CHARACTERISE THE TRANSITION

In order to identify ruptures according to didactics variables variations, we opted therefore for the survey of the values of the following variables, at the end of the secondary course and in the beginning of the academic one:

- VD1: The degree of formalisation in the domain of the analysis.

- VD2: The setting of validation, the limit algebra or the analysis one.

- VD3: The degree of generalisation.

- VD4: The number of news notions introduced in the limit environment.

- VD5: The type of tasks : heuristic or graphic or algorithmic.

- VD6: The choice of techniques, that are transparent or are about an amalgam.

- VD7: The degree of autonomy solicited, that we qualify by a ladder of routine.

- VD8: The mode of intervention of the notion, process statute or object one.

- VD9: The type of conversion between the semiotic representation settings.

Indeed, the choice of didactical variables' values determines the efficient didactical contract. Particularly, the values these variables take indicate the sharing of responsibilities between students and teacher in the mathematical knowledge. It imports therefore to present choices adopted at the considered levels, in order to analyse the carving of responsibilities and to be in position to surround some meaningful "modifications".

In every teaching institution – upper secondary school and University – we categorised the tasks listed in the limit environment at considered levels in several fields of observations. Every field permitted to mention some quantitative and qualitative information about values taken by each of the didactic variables mentioned.

Referring to this work, it appeared that the modifications of the values of these didactic variables constitute factors of global ruptures in the transition between school and University.

II. Didactic variables values and their consequences

II.1 FROM ALGORITHMIC WORK TO COMPLEX TECHNIQUES

Exercises and problems of the text books of the high school use most of the time algorithmic techniques that will then become a routine. However, sets of controlled works at the beginning of the University present an amalgam of complex techniques.

At the University students are confronted with activities of reasoning. These activities are very often centred on mathematical generalities, while their resolutions require the use of heuristic techniques as proving or conjecturing or reasoning by reductio ad absurdum or the research of counter-examples. These activities require a formalisation's work that is not usual at the high school.

At the beginning of the University, the phase of validation is elaborated in reference to the specific system of proof of analysis including the use of the formal definition of a limit with ε and η (Bloch, 2000), while at the high school, the work on validation emphasises the algebra of limits and functions are always given by algebraic expressions, so that no work is done about general properties.

At the same time, the number of techniques that are supposed to be mastered by students increases considerably, while students get not enough opportunities to mobilise these techniques. It appears that there is a requirement on students'

behalf to manage personally their difficulties: the complexity of the students' personal work increases considerably.

At the high school, students have almost no occurrence to take the decision of using a diagram or a graph in an heuristic way. However, this work could be very helpful in the resolution of integrative problems that involve the concept of limit. Maschietto (2001) points the functionalities of the graphic representations for the resolution of analysis problems at the beginning of the University. She notes that students get many difficulties to grab the opportunities offered by a diagram as an heuristic support during a phase of control or an exploration. At this point, it should be clear that at the beginning of the University students must develop such skills by themselves and without any indications. The teaching institution does not seem to make specific efforts for equipping students with abilities that permit the use of graphs.

At the entrance to the University, we note the apparition of exercises that support the structural conception of the notion of limit. These exercises are based on general conjectures; their resolution requires a perfect adaptation of students to the formal definition of the limit, whereas at the high school, the limit notion is conceived as a process. Its representations appear to be more susceptible of operational interpretations.

Finally, the utilisation that we make of the semiotic representations settings proves to be a support to detect no exploited potentialities at the University. Indeed, at University we note the disappearance of exercises that solicit a fruitful conversion between the setting of algebraic semiotic representatives and the graphic one; these tasks of conversion being rather usual and helpful at the high school, we are led to think that University does not take into account the more demanding and delicate stakes existing in the transition, because University does not take care of the previous knowledge that has been built in the upper secondary school.

To complete the survey, let us summarize the modifications that occur in the values of the considered didactic variables.

	End of the 2nd school course	Beginning of the academic course
VD_1 : Degree of formalisation	Weak	High
VD_2 : Setting of validation	Limit algebra	Analysis proof system
VD_3 : Degree of generalisation	Hopeless	High
VD_4 : Number of news notions introduced	In important rise	
VD_5 : Type of tasks	Algorithmic-Diagram	Heuristic

VD_6 : Choice of techniques	Transparent	Amalgam
VD_7 : Degree of autonomy solicited	Raised routine	Absence of routine
VD_8 : Mode of intervention of the notion	Process	Object
VD_9: Type of conversion between the semiotic representation settings	Algebra/Graphs	Algebra/Analysis

Before any illustration is given, it should be pointed out that the identification of the variables allows us to detect global ruptures at the passage from the secondary teaching institution to the superior one. The values given to these didactic variables are mutually exclusive. We can observe that almost all the variables change, and that the rate of change is considerable. Students are confronted with a global revolution in the required work and of their means of work.

II.2 EXAMPLES OF TASKS RELATED TO DIDACTICAL VARIABLES

What follows now is a very brief and by no means exhaustive presentation of the survey that permitted us to find the values of the didactic variables.

- We illustrate VD1 et VD3 with exercises of the University's first year:

Exercice1 de la série 3
Soit $(u_n)_n$ une suite à terme dans Z
Montrer que $(u_n)_n$ converge si et seulement si elle est stationnaire

Exercice1 de la série 4
En raisonnant par l'absurde montrer que :
Toute fonction continue sur un intervalle fermé borné est uniformément continue sur cet intervalle.

Exercice 4 de la série 3
Vérifier que chacune des propositions suivantes est vraie ou fausse :
Toute suite croissante non majorée tend vers $+\infty$.
Toute suite non majorée admet une suite extraite qui tend vers $+\infty$.
Toute suite convergente est bornée

All these exercises concern general properties, and require a formal proof.

- VD4:

When they enter the University, students can see that technologies that are taught are strictly Bourbaki's rules, with formalism and specific validation in the

theory of classic analysis. Above all, mathematical demonstrations they are asked must be complete. But at upper secondary school, the curriculum says:

"L'étude des limites n'est pas une fin en soi. Les différents théorèmes serviront surtout dans l'étude du comportement d'une fonction aux bornes des intervalles où elle est définie (dérivabilité à droite ou à gauche, branches infinies…). Il n'y a pas lieu de multiplier les exemples posés à priori. Les théorèmes relatifs à : la limite et l'ordre ; la limite d'une fonction composée, seront admis." (Programme officiel, 1998)

Then we can illustrate VD4 with new techniques and technologies introduced in University's first and second year, to operate on limits with the dimension of mathematical objects, with the formalism and the rules of analysis:

1^{st} year:	2^{nd} year:
1 Use formal définitions	1 Use formal définitions
2 Do a DL	2 Find that a sequence is a Cauchy one
3 Use counter examples	3 Identify adjacent sequences
4 Use reductio ad absurdum	4 Use extracted sequences
	5 Use counter examples
	6 Use reductio ad absurdum

VD5:
We illustrate VD5 by the percentage of heuristic tasks at the beginning of the University and the percentage of algorithmic and graphic tasks at the end of high school; and we give too exercises of the University where graphs can be useful.

At the upper secondary school, 52% of the tasks are algorithmic, and 33% are done with the help of graphs; heuristic tasks are about 14%.

At the beginning of the University, we inventory 37% of algorithmic tasks, no occurrence of graphic tasks and 63% of heuristic tasks with new techniques or technologies.

Example of exercise where a graph can help students to solve:

Solution of an equation as f(x) = x (limit of a sequence); or limit of a sequence with a parameter in the function, as (x_n): $x_0 = 1$ and $x_{n+1} = a \sin x_n + b$

- VD6:

At the University you can see that numerous techniques are taught, and in a same task students have to use an amalgam of techniques; at upper secondary school, a few techniques are very well identified and many exercises permit a work on these techniques. Students then are never surprised by the work they

have to achieve; at the University they have the responsibility of choosing the adequate technique, and very often an exercise requires not only one technique.

- VD7
Study of sequences of the type $u_{n+1}= f(u_n)$: at the University it is necessary to identify the smallest interval where the function decreases, whereas at the upper secondary school students just had to answer the questions and were given the good upper and lower bounds.

- VD8 :
We can see exercises using the object aspect – instead of the process aspect being used at the upper secondary school.
We think that the process / object transition is a efficient model in our classification because it allows to describe and categorize the usual tasks at upper secondary school versus University, and it permits to seize the conceptual "jump" students are led to:

- Achieve reification about the concept of limit;
- Gain the unifying formalism (definition with ε, N), and by this way generalise the notion of limit and realize the economy of this knowledge to do the usual tasks and overall the economy in thinking.

But indications about the process or the object can be given in an exercise or not: we identify exercises about the process, with indication or not, and exercises about the object, object being indicated or not. We noticed a few exercises that show different aspects – process indicated: PI, process with no indication: PNI, and similarly for the object.

Exercise 10 in page 81, in the secondary last year's text book: study the continuity of f in 0, PNI, process with no indication.

Exercise 7 in page 121 of the same text book, studying the derivability of with the rate of instantaneous variation: PI.

Exercise 6 of series 3, first year of the University: find the limit in 0 of: $x \times \sin(1/x)$, aspect OI (Object with indication).

Exercise 4 of series 3, first year of the University: 6^{th} question, prove that every convergent sequence is bounded, aspect ONI.

II.3 A NEW CLASSIFICATION: PI, PNI, OI, ONI This classification – PI, PNI, OI, ONI – proves to be useful in studying exercises, because it applies to the most part of them. And the results of our study is completely coherent: at the University, most exercises imply a PNI or an object (OI or ONI), while at the high school, almost all the exercises imply a indicated process. When coupled with the other didactical variables, one can see that students must get used to a completely new way of working with limits: limits as objects, with a high

formalism, many amalgamated techniques, no routines, a high level of generalisation, heuristic tasks, and representatives in the field of algebra or analysis, no more graphs, everything must be proved.

Conclusion

According to this study, an other important hypothesis is that the role of the teacher must become especially difficult in such an environment: How to lean on the maladjusted previous knowledge? How to bring students to take into account the new requirements? The teacher's role should be studied carefully in the following of the research.

Subject to an ulterior survey centred on the reality of the work in a class, our results tend to confirm that the transition between the secondary school and the University is accompanied by a modification in the mathematical game and the intentions. This situation leads to change the contract's rules, which is difficult to seize by students and to manage by teacher. This situation can be qualified as an important rupture.

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