A VARIOUS MILIEU FOR THE CONCEPT OF LIMIT : FROM DETERMINATION OF MAGNITUDES TO A GRAPHIC MILIEU ALLOWING PROOF

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Abstract : The mathematical work in the field of Calculus is usually very difficult for even good students when they are entering the University; one reason is that students never get the opportunity of an explicit preliminary phase of experimental character during which mental objects and related intuitions would be an explicit reference to the future formal explicitation of the knowledge. In this paper we aim at introducing some examples of the first heuristic work that teachers can propose to their students before these ones have to understand formal definitions and proofs of the main concepts of analysis. This work leads students to cope not only with intuitions and mental objects, but also with pragmatic proofs, as graphic ones.

A various milieu for the concept of limit : from determination of magnitudes to a graphic milieu allowing proof

Such that it is habitually conceived in many countries, the teaching of analysis does not include an explicit preliminary phase of experimental character during which mental objects and related intuitions would be an explicit reference : from 16 years, students have to assimilate at the same time phenomena associated to irruption of infinity and limits, and concepts, formal theories that express and develop them in mathematics. Are mental objects taken into consideration on the occasion of the exercises, compensating their absence during the elaboration of the theory ? This is, in general, not the case : one observes indeed too often an overemphasis on formal calculation of derivatives and primitives and on systematic studies of graphical representations of functions, with no context. A lot of teachings of analysis seem to be organized during their first year according to a central objective : to determine graphic features of analytic expressions (exercises called "variation of function"). Limits and derivatives are introduced almost in this only perspective.

On the other hand, the usual introduction at upper secondary school consists in looking at a few examples of functions and limits, noticing their properties and eventually reaching generalization in some implicit ways, without all the tools of formalisation. This is supposed sufficient as a first approach to the concepts of analysis, assuming that, later (at the university level), students will learn to prove and justify the properties introduced at this point. But at the university, teachers often complain that students do not show the proper abilities to prove and that they use graphs and equations as if they were some kind of "labels" for functions, rather than material means for expressing concepts and tools for proving; and that they remain on intuitive conceptions of limits instead of grasping adequate formal tools for proving.

In accordance with the Theory of Didactical Situations (G. Brousseau, 1997), we think that, in a teaching device concerning the concept of limit, the epistemological analysis must take into account the epistemological obstacles linked to mental objects and we think that this teaching must force students to prove. The project we expose here tries to build learning situations starting from the corresponding conceptions, and to lead on to situations for proving and reasoning about mathematical statements on functions and limits.

This project illustrates both concepts of milieu and system. The concept of "milieu", introduced by Guy Brousseau, allows to think not only of the setting that is used, but of the tasks and the responsibility (teacher's or students') for the formulations and the proofs. Brousseau speaks of the *milieu* as an organization of the teaching:

> The student learns by adapting herself to a *milieu* that generates contradictions, difficulties and disequilibria, rather as human society does. (Brousseau,1997)
> The milieu is the system opposing the taught system, or, rather, the previously taught system. (Brousseau 1997: 57).

By "system" he means a teaching-oriented organized system; and it is organized by the teacher, or the researcher, to obtain mathematical work from students. Brousseau describes the different stages of an adidactical situation (action, formulation, validation, see Brousseau 1997: 8-17) and defines the three different *milieus* according to the stages: material milieu, objective milieu, milieu of reference (Brousseau 1997: 248). The material milieu is made of material "things" to act with. In the case of the graphic approach for functions, those things are drawings of graphs (when we say "things" we mean that, for the students, they do not necessarily represent mathematical objects, or at least coherent ones). The objective milieu includes heuristic procedures which must lead students to formulate mathematical properties in the milieu of reference. Then the didactical situation allows the teacher to declare the knowledge that has been aimed at.

1. From the identification of epistemological obstacles to the design of fundamental situations

Several researches have also brought to the fore that the learning of analysis stumbles on epistemological obstacles linked to mental representations of students (cf. e.a. D.O. Tall and al. (1981), A. Robert (1982), C. Hauchart (1985), A. Sierpinska (1985), M. Schneider (1988)).

1.1. THREE RELATED EPISTEMOLOGICAL OBSTACLES

According to our epistemological analysis, here are some examples of three related obstacles.

<u>Example 1 : some students seem to discard the concepts of instantaneous flow and velocity</u>

Some students would discard the concepts of instantaneous flow and velocity. They invoke the necessity to have a minimal volume to obtain a flow : *"To obtain a flow, it is necessary that it remains a small volume"*, or they invoke the necessity to have a non zero space to determine a velocity : *"If there is no time, the moving body is at rest"*. They emphasize the fact that every measure requires a minimum of time : *"That does not exist, it is not possible to measure it, because during the time one looks at ones watch, the time has already flowed"*. These students seem to deny to mathematics concepts the possibility to circumscribe with some precision things that appear to them, to some extent, beyond senses and measures (for more details, see M. Schneider, 1992), expecting unconsciously that mathematics extend their first perception of an illusory "sensible world", (that is a world they think they are able to apprehend by their senses), This conception characterizes a *positivist view* about mathematics.

<u>Example 2 : do rectangles that reduce into segments lead to an area under a curve ?</u>

Is the area under the curve $y = x^3$ *between abscissas 0 and 1 (defined in an intuitive sense) equals 1/4 exactly, i.e. the common limit, as n tends to infinity, of the expressions*

$$(1 - 2/n + 1/n^2) / 4 \quad \text{and} \quad (1 + 2/n + 1/n^2) / 4,$$

representing respectively the sum of the areas of the n rectangles shown in Fig. 1 *and* 2 *?*

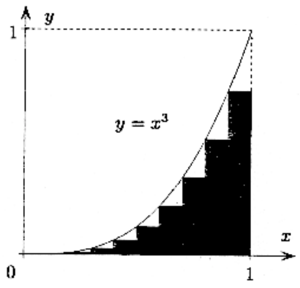


Fig. 1

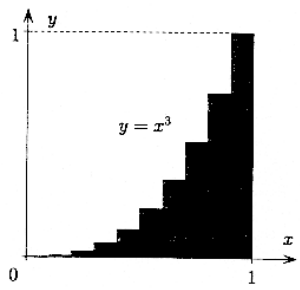


Fig. 2

Several students claim that this is only an approximate result, because of the following alternative :

- as long as the rectangles have a certain thickness, they do not entirely fill the surface or they exceed it : small "triangles" remain to be filled or removed;
- when they reduce to segments, one badly sees how to obtain an area differing from zero and infinity, by adding their zero areas or their lengths.

Once formulated, this alternative is a dead end for the imagination of the other students and sometimes of their professors.

We can interpret this reaction in the following manner : students conceive the "limit" on the level of their visual imagination of magnitudes where rectangles narrow until becoming true segments, instead of considering the limit of the concerned sequence in the numerical sense. After what, they seem to come back to the numerical field in an attempt at interpreting 1/4 as the sum of measures of the remaining segments. Therefore their imagination seems to be caught in a dead end, because unduly deviated from numbers to magnitudes (for more details, see. M. Schneider, 1991b and 1991c).

Example 3 : a deceiving cutting up of revolution solids into radial sections

A same mental slipping from magnitudes to their measures explains why several students expect that two volumes of revolution have the same ratio as the areas of the surfaces generating them. For example, the straight cone, generated by the revolution of a triangle, would equal half of the cylinder which, having the same base and height, is generated by a rectangle of double area, see Fig. 3. Such slippings constitute an epistemological obstacle, the *obstacle of heterogeneous dimensions*, identified by M. Schneider, 1991b.

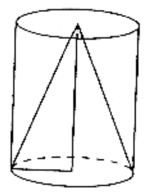


Fig. 3

This obstacle of heterogeneous dimensions includes the *geometrical obstacle of limit*, studied by A. Sierpinska (1985) about tangent lines and illustrated by example 2 below. It depends also upon a positivist view about mathematics evoked in example 1.

After these examples, an observation, important for our purpose, is necessary: difficulties that we have exhibited persist often among students of the secondary school even after a first course of analysis, and they are not rare, neither among university students in mathematics nor among professors teaching mathematics at secondary school. Moreover, several of the intuitions described above have been shared by mathematicians in the course of history. For example, the reticences of students about instantaneous flow and velocity remind us of those of Berkeley (1734) against the concept of "ultima ratio" proposed by Newton (17^{th}-18^{th} Century) : *"But how can we conceive a velocity by means of such limits? A velocity depends on the time and the space, and cannot be conceived without them [...] "*.

The quasi universal presence of analogous difficulties among mathematicians in the history, today among students, and their persistence, indicate the presence of epistemological obstacles in accordance with the viewpoint of G. Brousseau (1983) : *"Obstacles of epistemological origin are those which one cannot, neither does not have to escape, because of their constitutive role in the aimed knowledge. One can find them in the history of the concepts themselves[...]"*.

1.2. OTHER DIFFICULTIES DUE TO THE FORMALISM

We also can notice other difficulties, due to the mathematical formalism. The system that mathematicians use to prove and to control mathematical assertions has been built through history, and was adapted during the whole 19th century and a large part of the 20th. A mathematical work can only be achieved using semiotic tools that are more or less convenient to do the work and are the subject of adjustments: let us think to the ***o*** and ***O*** of Bourbaki… Building situations for learning the concept of limit must then take into account the kind of semiotic representatives that is used; and we must not forget that a proper mathematical knowledge, especially including proof, is built only if the selected semiotic representatives and the milieu allow adequate reasoning, *and* if students can seize these tools of control.

1.3. FUNDAMENTAL SITUATIONS ABOUT CONCEPT OF LIMIT INSPIRED BY THESE EPISTEMOLOGICAL OBSTACLES OR THE MILIEU OF MAGNITUDES

Inspired by lessons of the history and in the light of learning difficulties persistent among initiates, we propose a teaching taking into account the intuitions related to infinity and giving the students the opportunity to express their mental objects and to refine them little by little for overcoming the underlying epistemological obstacles. The selected milieus allow finally reasoning and proving with formal mathematical symbols.

Motions and velocities

A first possible stake is to approach the instantaneous rate of change in problems of constrained velocities. Such is the following : *A pump fills up a conical vase (Fig. 4). It is regulated in such a way that the level of the water climbs regularly by one centimeter per minute. The angle at the vertex of the cone has an amplitude of 90°. Until when will the flow of the pump be inferior to 100 cm^3/min ?*

In this problem, as in some others, the fact that a magnitude (here the level of water) varies with constant velocity induces the intuition that the other magnitude (here the volume V) cannot vary with constant velocity (since the width of the vase increases). Hence, in order to evaluate the velocity of variation of the volume, the necessity to cut the time into smaller and smaller intervals.

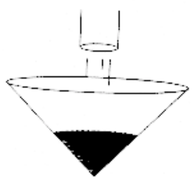


Fig. 4

The average flow of water on the interval $[t, t + \Delta t]$ reads

$$\frac{V(t + \Delta t) - V(t)}{\Delta t}$$

or, in the present case,

$$\pi t^2 + \pi t \, \Delta t + \pi (\Delta t)^2/3. \qquad (1)$$

The idea arises then to reduce Δt as much as possible, which is here equivalent to removing the terms containing Δt in expression (1). The expression obtained in this way, that is πt^2, is called *instantaneous rate of change* of *V* at this instant *t*, the limit meaning thus in practice "to put $\Delta t = 0$".

In order to answer the hesitations of the students facing the boldness of this calculation and their reticence about the concept of instantaneous flow (see example 1 of the section 1.1), one proposes a thought experiment susceptible to convince them that the result obtained in this way is effectively exact. It consists in putting beside the conical vase a cylindrical one with a base of 100 cm^2 (Fig. 5) and filling up both using pumps regulated in such a way that the levels of water climb in both regularly and simultaneously by 1 cm/min. The pump that fills up the cylinder has obviously a constant flow of 100 cm^3/min. The other pump will have an outflow slower than the first as long as the cone is narrower than the cylinder, and faster afterwards. Consequently, the two pumps will have the same flow of 100 cm^3/min. at the precise moment where the surface of water in the cone measures 100 cm^2, which is equivalent to equating the instantaneous flow πt^2 to 100.

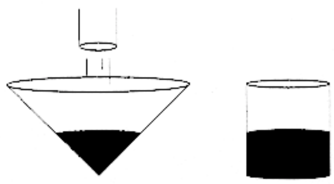


Fig. 5

After, the teacher proposes to his pupils a debate about text in which Berkeley express his reticence about the concept of « ultima ratio », the equivalent of instantaneous velocity.

Epistemological and didactic stakes of this problem are analysed in M. Schneider (1992). She shows how a various milieu offers here some moments of devolution : the problem and its didactical variables, one epistemological obstacle, the previous experience of pupils about breaks of didactical contract and the efficiency or the limits of algebraic calculation, forbidden things in this calculation, debate between pupils and so one.

From the proof of an area calculation to the concept of limit

The two sequences approximating the area under the curve $y = x^3$ in example 2 below have the same limit, namely 1/4. But, as seen above, all students are not convinced that this limit is the exact area sought for. Hence the interest to propose a proof through which something of the formalized concept of limit is built.

To convince students that the area under $y = x^3$ equals exactly *1/4*, we formulate a proof <u>ad absurdum</u>, inspired by the exhaustion method and relying on intuitive evidence : on the one hand, this area is bounded below and above by the sum of areas of Fig. 1 and 2 ; on the other hand, a real number corresponds to each considered area. The area under $y = x^3$ cannot equal $1/4 + \varepsilon$, with ε as small as we please, for by subdividing the interval into a sufficiently large number of segments, we may insert the sum of circumscribed rectangles between *1/4* and $1/4 + \varepsilon$ (Fig. 6). Hence the contradiction : the area sought for is larger than one of its approximations from above. One shows analogously that the area can not equal $1/4 - \varepsilon$.

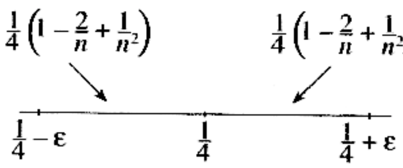


Fig. 6

As shown by M. Schneider (1988), this proof points to the (ε, N) concept of the limit of a sequence. Namely, it reverses the order of enunciation of the asymptotic behaviour as spontaneously used by the students : instead of beginning, as they do, by the behaviour of the subscript *n* and proceeding then with the corresponding behaviour of the sequence, one looks first at the terms of the sequence and thereafter at the subscripts. Further, the choice of the latter is subordinated to the choice of the a_n : one likes to have $(1 + 1/2n + 1/n^2) / 4$ smaller than $1/4 + \varepsilon$ for given ε, one looks for the *n* values satisfying this condition. Further again, the quantifiers $\forall$ and $\exists$ and their classical presentation : $\forall \ldots \exists \ldots$ appear as a kind of watermark through this proof. Namely, on the one hand, one has to verify the inequalities

$$(1 + 2/n + 1/n^2) / 4 \;<\; 1/4 + \varepsilon \text{ and } (1 - 2/n + 1/n^2) / 4 \;>\; 1/4 - \varepsilon$$

whatever the value of ε. On the other hand, the contradiction appearing through this proof follows already from the **existence** of a value of *n* fulfilling a given condition .

Of course, if the limit concept is mobilized here, it is in an implicit and sketchy way only. One is far from its symbolic formulation. But what appears important in this proof ad absurdum is that it makes one feel what will be expressed by the technical formulation of the limit concept, and leads to realize the instrumental role to be played later by this formulation. This proof rests in fact on the possibility to get at the same time

$$(1 + 2/n + 1/n^2) / 4 \quad \text{and} \quad (1 - 2/n + 1/n^2) / 4$$

as close as one wishes to *1/4* , starting from a given rank.

2. Graphs with asymptotes : a graphic milieu for another point of view about limits

It is then necessary to conceive another panel of tasks about limits, as to build an adequate assortment leading to sufficient various work about the notion. The context of functions seem unavoidable to get a sufficient point of view about limits, but the difficulties students encounter with algebra may narrow the possibilities of working. A study showed that the graphic setting provides us with many interesting tasks about functions (see Bloch 2003), but that it cannot be used alone if students are expected to engage with proving. It is useful for conjecturing, and, coupled with convenient formal tools, it allows for a large choice of problems on functions, limits and asymptotes. The aim is to organise tasks about graphs, equations, and limits of functions, as to really work on mathematical knowledge, as Slavit (1997) recommends it.

We have used the Theory of Situations, to develop a teaching approach of functions and limits within a graphic milieu for 17 year-old science students.
What do we mean by a "graphic milieu"? The idea of using a setting – like the graphic one – is not sufficient to give an account of the mathematical work being done by either the teacher or the students. Using the same setting, the organization of the work could be with the teacher just "showing" limits of functions (actually, on graphs) in a sort of display; or students could have some real work to do about properties of functions through graphic study. With a proper graphic milieu, students are led to work on graphs, and to discuss properties of functions, including limits and asymptotes. To prove they use paths (see Bloch 2003): direct paths look like the horizontal test line.

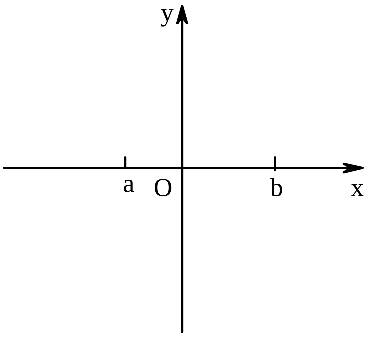
The aim is to engage students with drawing graphs of functions, studying their properties, as well as producing and questioning statements about functions. One work consists in studying properties of functions that are linked with the order on **R**, such as being bounded, increasing or decreasing. This leads students to make the difference between a condition like: "f is bounded by f(a) and f(b)", which is expressed with one universal quantifier:

$$\forall x \in [a, b] , f(a) < f(x) < f(b)$$

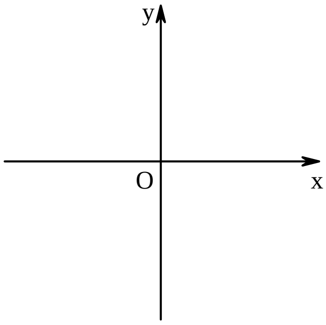
and a condition like: "f is increasing on [a, b]" , which requires *two* universal quantifiers:

$$\forall x \in [a, b] , \forall y \in [a, b] , \text{if } x < y , \text{then } f(x) < f(y).$$

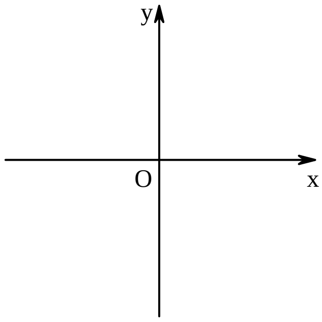
In the experiment, there was a discussion between students, some of them being sure that the first condition meant the same as the second one, that is, that a bounded function (like the one in the first space in Figure 7) is necessarily increasing. A counter-example had to be produced by some students, as a curve which satisfies the condition but is not increasing. This was work on the *meaning of quantifiers,* that is, on the *control system* of mathematical assertions in the field of analysis**.** The same graphic allows a work about continuity in *a*.



∀x / x < a or x > b, f(x) < f(b) and ∀x / a < x < b , f(a) > f(x) > f(b)



On **R**, f is bounded from above but not from below



On]0, 1[f is not bounded from above

Figure 7

The second case in Figure 7 may lead to horizontal asymptote, which does not seem to perturb the students, since the horizontal asymptote appears as an external above "limit", but with the sense of a limit not to be exceeded by *y*.
The condition of the last case in Figure 7 is quite difficult, since students at this level are not acquainted with curves with asymptotes, and even less with more pathological functions. In the experiment, they were trying to express the condition with quantifiers. First, they tried

$$\forall M \in \mathbf{R}, \forall x \in]-2, 1[, f(x) > M$$

but they soon discovered that such a function would be hard to imagine: they could not draw any graph with this condition. So they found that a better condition is

$$\forall M \in \mathbf{R}, \exists x \in]-2, 1[, f(x) > M$$

and tried to find a convenient function, but it is not easy in a graphic milieu, because it is not possible to build an unbounded function that could be seen in a convincing way (the window is limited). The teacher had to give them a function with a formula, $f(x)=1/x^2$, so they could verify with algebraic calculation that it is not bounded. But this verification was not easy for them. This shows that in some cases, the graphic work must be combined with the algebraic one to enhance understanding.

When trying to prove that the function $f(x) = 1/x^2$ is not bounded in the interval [0,1] students used different methods: some of them tried to show that it can exceed 60, because for them "60 is a rather big number". Others showed that it exceeds 10^{98} because it is nearly the biggest number of their calculator; and a few said: "Let us do it for every M, so as to be sure". They wrote the solution they had found:

$$\forall x , \text{ if } x < 1/\sqrt{M} \text{ then } f(x) > M$$

The formulation above is not correct because: 1) they did not see that to find one x is sufficient, and they wanted that every x fits; actually in this case, one finds a whole interval when looking for one x, and it was difficult for them to select one value of the interval. 2) Moreover, the use of quantifiers is rather approximate at that level in the French system, and formulations such as: "For every x, if x is in the interval …, then …" can be found in the text books. Then students just reproduced such a formulation.

But they ran into a problem because the condition $x \in [0,1]$ is not satisfied for every x in case M <1 and $x < 1/\sqrt{M}$.

Then instead of choosing some $x \in [0,1] \cap [0, 1/\sqrt{M}]$ – the intersection is not empty – they said: if M<1 the property is trivial, so let us do it for M>1 only, and we are sure that this way $1/\sqrt{M} < 1$ and we stay in the interval [0,1].

We can identify this reasoning as typical of analysis; it is a reasoning by a sufficient condition, even if it is not the expected sufficient condition (to find *one x* in the intersection is sufficient and always possible). So in this case the graph helps to conjecture that there could exist unbounded functions in a bounded interval, but the convincing proof takes place in the algebraic and analytic settings.

There are three very different levels of "proof" in this work:

- on the first level, the proof that there exists an *x* such that $f(x) > 60$, has the status of nothing more than a calculation;
- on the second level, 10^{98} appears as a «generic number» to become convinced that *f* can exceed a «very big» number, so it is not bounded;
- on the third level is a real analytic proof.

At the end of this work, the link with limits is not evident and must take place in the institutionalisation's phase : $\lim_{x \to 0} 1/x^2$ is infinite.

We can see that this work is a key point of a graphic approach, because it leads to the target knowledge – analytic thinking – and engages students with properties of functions and limits of them: what are the functions that satisfy a property, what is the meaning of a property such as to be bounded, or *not* bounded and get an asymptote. It also leads them to use the formal setting, such as quantifiers, and to discuss their number and order, that is, to give oneself formal means for controlling the validity of assertions. It eventually leads students to complete analytic reasoning by sufficient condition. The milieu of the third phase is thus part of a *learning* situation, one that permits to validate and argue; according to Brousseau, a learning situation in mathematics is one in which mathematical knowledge is discussed, validated and the process of institutionalisation begins. (See Brousseau 1997: 248).

Conclusion

Intuitions linked to the concept of limit are very different from one context to another. If a velocity, an instantaneous rate of change and a slope of a tangent are equivalent from a mathematical point of view, they are different from an epistemological and contextual one. To cancel *x* in the calculation of a derivative does not perturb in the same way than to remove terms containing $1/n$ or $1/n^2$ in the calculation of an area under a curve, despite the fact that both cases concern the same limit calculation: indeed, $1/n$ and $1/n^2$ can become zero only if *n* "reaches" infinity. It is more difficult to accept that one can obtain exactly an instantaneous velocity or an curvilinear area by suppressing terms that one obtains an horizontal asymptote of a function in the same conditions because the asymptote is **external** to the curve in several cases.

By opting as usual a priori for a mathematical and thus unifying point of view, that is by beginning the teaching of analysis with limit and continuity concepts, one denies these differences without giving students time enough to live with them and to realize their common essence. Sooner or later, questions, specific difficulties to each of the evoked contexts reappear (if nevertheless the opportunity happens), but sometimes too late to take them into consideration. We think that students must get some experience of different contexts for limits, and test their conceptions; but they must study situations that permit validation and a real mathematical work. Situations built from analysis of obstacles and situations in a graphic milieu both allow this kind of work, in a rather satisfying progression.

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