Can we do something to improve the teaching of first-year calculus?

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After teaching mathematics for many years I have come to the conclusion that the above question has an affirmative answer. As is often mentioned, teaching is an art rather than a science, and I do not presume to have the cure to all the ills that plague mathematics education. However, it is my belief that something can be gained by paying special attention to five basic guidelines when teaching calculus. They are as follows:

1. Try to strike a balance with regard to what to prove and what to accept without proof.
2. Convey the idea that sometimes there is more than one way to solve a problem.
3. Discuss significant applications in the classroom, not relegating them to the end as optional materials.
4. Place the subject in a historical perspective whenever possible.
5. Use technology to supplement mathematical learning, not to supplant it.

In recent times the mathematical community has addressed the problem of how to improve teaching of calculus (Steen, 1987; Tucker, 1990; Leinbach, Hundhausen et al., 1991; Schoenfeld, 1997), and teaching of mathematics in general (Krantz,1999), thus showing their concern about this complex issue.

PROOFS

Let us agree that in every calculus course, especially for an audience of mathematics, computer science, natural sciences or engineering students, some basic facts have to be proven. Not everything needs a proof, of course. For instance, in first semester calculus three propositions, all of them unexpected, can be proven in great detail assuming the "expected" limit laws, namely:

1. Differentiation of the product of two functions.
2. A particular case of the chain rule, for example the derivative of the composition of the square root and a differentiable function.
3. The derivative of the inverse sine function, on the basis of the chain rule once we accept that the inverse of a monotonic differentiable function is differentiable too.

My students know that on every test a question will deal with variations of the above-mentioned propositions; for example, the derivative of the quotient of two functions, the derivative of the composition of the second power function and a differentiable function, or the derivative of the inverse cosine. This way I hope to foster understanding rather than mechanical reproduction of given proofs.

We state precisely but rely on geometric intuition in the case of the mean value theorem, the max-min theorem, the intermediate value theorem, and the mean value theorem for integrals (MVTI). The latter is sometimes proven as a corollary of the fundamental theorem of calculus (FTC) and the mean value theorem (Stewart, 2001, p. 474), but this is not really necessary because it amounts to using a non-evident theorem (FTC) to prove an evident theorem (MVTI). On the contrary, the acceptance of MVTI on the basis of its geometrical interpretation for positive functions leads to Cauchy's proof of FTC, as we will see later. In any case, MVTI can be rigorously proven quite rapidly starting from the max-min theorem and the intermediate value theorem.

As expected, a considerable amount of our work in first-semester calculus is concerned with calculating and using derivatives, and solving applied problems rather than building a rigorous foundation for calculus.

In second-semester calculus we analyze and prove only five propositions:

1. The fundamental theorem of calculus, namely $\frac{d}{dx}\int_a^x f(t)dt = f(x)$, and the evaluation theorem ($\int_a^b f = [F(x)]_a^b$, where F is any antiderivative of the given continuous function f).
2. Formula for integration by parts.
3. Formula for change of variables.
4. Comparison test for series of positive terms.
5. Ratio test.

Experience has shown us that there isn't much time left for other proofs in a semester loaded with many techniques and mathematical ideas, as well as applications. It should be emphasized that we carefully state and make plausible all the other results needed in calculus of one variable.

Three decades ago R.P. Boas addressed the issue of what to prove and not to prove when he wrote (Boas, 1971): *In any case, only so much time is available. In order to make the best use of it, I claim that the teacher of calculus would do well to follow the lead of the experimental scientist: let him give proofs when they are easy and justify unexpected things; let him omit tedious or difficult proofs, especially those of plausible things. Let him give easy proofs under simplified assumptions rather than complicated proofs under* *general hypothesis. Let him by all means always give correct statements, but not necessarily the most general ones that he knows.*

In the seventies there was a tendency to prove too many things in first-year calculus. Nowadays there appears to be a growing tendency not to prove anything at all, although several valuable ideas have been put forward to bring proofs back to the classroom (Young 1996, Olson 1997, Tucker 1999).

MORE THAN ONE WAY

Many students come to college with the idea that for every mathematical problem there is a unique path to the solution. This belief hinders the search for their own path and conveys the wrong message. Calculus offers many opportunities to dispel any notion about unique paths to all problems. For example, when asked to calculate $\int x\sqrt{x+1}$ it is possible to find the answer either by using substitution or integration by parts.

A fertile terrain for the visualization of multiple ways of solving problems is the theory of series. For instance, the series $\sum \frac{1}{n^2}$ can be determined to be convergent by comparing it with $\sum \frac{1}{n(n+1)}$ or by using the integral test.

Some proofs can be given using different approaches. For instance, the evaluation theorem can be proven independently of FTC (Stewart, 2001, p. 370) or as a corollary of it, a fact that is not often emphasized: Let f be a continuous function and F any antiderivative of f. Defining $G(x) = \int_a^x f(t)dt$ we can conclude that $G'(x) = f(x)$ thanks to FTC. Since F and G are antiderivatives of the same function it follows that they differ by a constant, i.e. $F(x) = G(x) + k$ for some constant k; in particular $F(a) = G(a) + k$. But $G(a) = 0$, thus $k = F(a)$. On the other hand $F(b) = G(b) + k =$ $\int_a^b f(t)dt + F(a)$, therefore $F(b) - F(a) = \int_a^b f$.

With regard to applied problems, a dual vision can often be used. For instance, Newton's law of cooling states that $T'(t) = -k(T(t) - T_m)$, where T_m is the constant temperature of the refrigerating liquid or gas. The differential equation can be solved in two different ways, either as a separable equation or a first order linear equation. In general, any differential equation of the type $y' + ay = b$, with a and b constants, can be solved either by writing $\frac{y'}{b - ay} = 1$ and then integrating or by multiplying both sides of the equation by e^{at} converting it to $(e^{at}y(t))' = (\frac{b}{a}e^{at})'$; the latter is solved through standard techniques of differential calculus (no integration process is needed). Students should be aware that many important first order differential equations cannot be solved by both methods: $LI'(t)+RI(t)=E_0\cos wt$ (RL electric circuit with electromotive force $E_0\cos wt$) is linear but not separable, while $P'(t)=kP-\frac{k}{K}P^2$ (logistic growth equation) is of the separable type but evidently not linear.

Several simple problems about maxima and minima can be solved through a calculus approach and also by using algebra exclusively: A farmer, who has b ft of fencing, wants to build a rectangular field bordering a river, with no fence along it. How could the farmer obtain the largest area? Letting x and y denote the dimensions of the field we get the equation $2x+y=b$. Consequently $A(x)=x(b-2x)$. The usual procedure is to take the first derivative and equate it to zero. Another path is to avoid calculus altogether and solve the problem through algebra. Indeed, by completing squares we have

$$A(x)=-2x^2+bx=-2(x^2-\frac{b}{2}x)=-2((x-\frac{b}{4})^2-\frac{b^2}{16})=-2(x-\frac{b}{4})^2+2\times\frac{b^2}{16}$$

It is clear that by selecting $x=b/4$ the area function will adopt its maximum.

Problems about equations of tangents to conics can also be solved with and without the machinery of calculus. Suppose that we want to find the equation of the two lines that pass through (5,0) and are tangent to the ellipse given by the equation $\frac{x^2}{9}+\frac{y^2}{4}=1$. From a calculus perspective we apply implicit differentiation and obtain $y'=\frac{-4x_0}{9y_0}$, where (x_0,y_0) is the point of tangency. However, since the points (x_0,y_0) and $(5,0)$ lie on the tangent line we can conclude that its slope is $\frac{y_0}{x_0-5}$. Therefore $\frac{-4x_0}{9y_0}=\frac{y_0}{x_0-5}$, which in turn leads to $9y_0^2=-4x_0^2+20x_0$. On the other hand $y_0^2=4-\frac{4}{9}x_0^2$ because (x_0,y_0) lies on the ellipse. Solving these two equations in two unknowns we get $x_0=\frac{9}{5}$ and $y_0=\pm\frac{8}{5}$. Consequently $m=\frac{\pm\frac{8}{5}-0}{\frac{9}{5}-5}=\mp\frac{1}{2}$, so the equations of the tangent lines are $y=\mp\frac{1}{2}(x-5)$.

Surprisingly, a non-calculus approach is a viable option to solve the above-mentioned problem. Indeed, let $y=m(x-5)$ denote the equation of any line that passes through $(5,0)$. Since this line has to touch the ellipse we might as well analyze the equation $\frac{x^2}{9}+\frac{m^2(x-5)^2}{4}=1$, which is equivalent to $(4+9m^2)x^2-90m^2x+(225m^2-36)=0$. This quadratic equation has to have its discriminant (Δ) equal to zero in order to make sure that the line touches the ellipse at just one point. Hence

$$0=\Delta=8100m^4-4(4+9m^2)(225m^2-36),$$

that is to say $0 = -2304m^2 + 576$. Therefore $m = \pm\frac{1}{2}$, thus the equations of the two tangent lines will be $y = \pm\frac{1}{2}(x-5)$. The method portrayed in the previous lines can be applied to any ellipse and any point lying on either the horizontal or vertical axes, or for that matter any point outside the given ellipse. A similar analysis can be employed when dealing with any conic (Baloglou and Helfgott, 2004).

We do not want to give the impression that for every standard calculus problem there is a non-calculus approach of similar difficulty, rather we should stress that for many problems the machinery of calculus is the best available option. For example, if we accept Fermat's principle of least time then Snell's law of refraction has to be true; many authors prove this implication using calculus. The non-calculus approach is worthwhile from a historical perspective, but it requires considerable more work (Helfgott and Helfgott, 2002). Moreover, quite often calculus is indispensable because a great deal of mechanics, electricity, magnetism, and optics depend on calculus, as well as many other areas inside and outside mathematics.

APPLICATIONS

Most calculus textbooks provide several real-life applications. Besides simple problems about kinematics, optimization, work, hydrostatic force, radioactive decay, and the like, some time should be set aside for an in-depth discussion of one or two applications per semester, where the model is first built from principles of physics and a clear distinction is made between physical ideas and mathematical developments. Nice examples of this sort are parabolic mirrors (Kline,1967,pp. 88-92), suspension bridges (Hahn, 1998, pp. 257-264), and the shape of the catenary (Simmons 1985, p. 716); the latter was one of the first important standing problems that were solved with the help of calculus.

Moreover, a short introduction to chemical kinetics can provide a proper setting for discussion of the scientific method, an important concept that we should impart to our students as early as possible. This method, in its modern formulation, can be traced back to Galileo Galilei in the early 17th century. Scientists usually work in three stages (Frank, 1957, p. 142):

1. *Setting up principles.*
2. *Making logical conclusions from these principles in order to derive observable facts about them.*
3. *Experimental checking of these observable facts.*

For instance, let us consider a chemical reaction $A + B \to products$, with k as the parameter of the reaction. Based on the law of mass action, chemists would put forward a model governed by the differential equations $A'(t) = -kA(t)B(t)$, $B'(t) = -kA(t)B(t)$ with initial conditions $A(0) = a$, $B(0) = b$. At the outset it is not known whether this is a valid model. It could well be the case that $A'(t) = -kA^2(t)B(t)$ or some other differential equation. We need to obtain mathematical consequences that can be compared with a table of values of $A(t)$ or $B(t)$ (at different times), obtained through experiments. Indeed, since $A'(t) = B'(t)$ we can ascertain that $A(t) + p = B(t)$ for some constant p. In particular $A(0) + p = B(0)$, thus $p = b - a$. Therefore $\frac{A'(t)}{A(t)(A(t) + b - a)} = -k$. If $a \neq b$, using partial fractions and a simple process of integration we reach the expression

$$\frac{1}{a-b}\ln\frac{bA(t)}{a(A(t)+b-a)} = kt$$

So, if we place the left side of this equality on the vertical axis and time on the horizontal axis, our ordered pairs will gather around a line that passes through the origin. We validate the model if this prediction is in agreement with data. Thereafter the value of k is calculated as the slope of the regression line, also called the line of best fit. We could find the explicit solution of the differential equation, namely $A(t) = \frac{a(b-a)a^{-k(b-a)t}}{b - ae^{-k(b-a)t}}$, but this is of little use in the validation process; after all, it is much easier to work with a line rather than an expression that involves exponentials. If $a = b$ we get $\frac{1}{A(t)} = kt + \frac{1}{a}$ without much effort, and a validation process can be performed with $1/A(t)$ on the vertical axis and time on the horizontal axis.

Physics is the main source of calculus applications, but let us not forget that chemistry is particularly suited for work with data, in the sense that among several competing models we have to choose the one that is most in agreement with experimental results.

It seemed to me then, and still does now, that the teaching of calculus is the natural vehicle for introducing applications, and that applications give the proper shape to calculus; they show how, and to what end, calculus is used.

(Lax, 1986)

HISTORY

We could develop a course from a systematic historical perspective, but this would only work with very motivated students. It is more realistic to use history as a pedagogical device here and there, employing modern notation and the genetic method (Mosvold, 2003) whenever necessary, to provide a cultural context and to impress upon students the idea that crucial advances occur by successive stages. An example of the latter assertion goes back to the early 17th century when mathematicians tried to calculate the area under the curve given by the equation $y = x^k$, k any natural number bigger than one, between

0 and b > 0. Archimedes had already solved the case k = 2 by noting that $\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$, so $\sum_{i=1}^{n}(\frac{ib}{n})^2\frac{b}{n} = \frac{b^3}{6}(2+\frac{3}{n}+\frac{1}{n^2})$. This is an approximation to the area under $y = x^2$, where we have added the areas of n rectangles of width b/n and height $(ib/n)^2$, $1 \le i \le n$. As n becomes bigger and bigger we will approach the number $b^3/3$, which can be taken as the value of the area. Around 1630 Cavalieri used the formula $\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$, known to Arab mathematicians as early as 1010, to reach the answer $b^4/4$ when $k = 3$. A similar procedure allowed him to solve the case $k = 4$ and subsequently for the natural numbers $k = 5,...,9$; he found that the answer was $b^{k+1}/(k+1)$. This was quite an accomplishment because it is not easy to find closed formulas for $\sum_{i=1}^{n} i^k$ when $k > 3$. Fermat, around 1650, solved the problem for any positive integer k, using a different and very clever strategy (Toeplitz, 1963, pp.53-55) that involved dividing the interval [0,b] in subintervals of different length. Finally, the problem became quite easy after Newton and Leibniz created what we know as calculus in the last quarter of the 17th century; we have $\int_0^b x^k = \left[\frac{x^{k+1}}{k+1}\right]_0^b = \frac{b^{k+1}}{k+1}$ since $\frac{d}{dx}\frac{x^{k+1}}{(k+1)} = x^k$.

History is a guide that reminds us that quite often mathematicians proceed from the particular to the general, as in the development of the fundamental theorem of calculus. FTC was first understood when dealing with continuous monotonic functions (Toeplitz, 1963, p. 97); thus this version has to be learned by students at the beginning. The monotonicity condition was dropped later on; as a matter of fact, an acceptable proof of FTC did not appear until 1823, in Cauchy's works (Fauvel and Gray, 1987, p. 570), more than a century after calculus was invented. This is a proof that we could accept today with some modifications at the end, replacing infinitely small increments by a limit process: Fix any x, $a < x < b$. For every $h > 0$ we have

$$F(x+h) - F(x) = \int_a^{x+h} f - \int_a^{x} f = \int_x^{x+h} f.$$

By the mean value theorem for integrals it follows that $\int_x^{x+h} f = f(c(h))h$ for some number $c(h)$, where $x \le c(h) \le x+h$. So $\frac{F(x+h)-F(x)}{h} = f(c(h))$. Thus

$$\lim_{h\to 0^+} \frac{F(x+h)-F(x)}{h} = \lim_{h\to 0^+} f(c(h)) = f(\lim_{h\to 0^+} c(h)) = f(x)$$

(note that we have used the continuity of f at x and the fact that the squeeze property of limits implies that $\lim_{h\to 0^+} c(h) = x$). In summary, we have proven that $F'_+(x) = f(x)$. In a similar fashion we can prove that $\lim_{h\to 0^+} \frac{F(x-h)-F(x)}{-h} = f(x)$, so

$F'_{-}(x) = f(x)$. Since the left and right derivatives of F are equal to $f(x)$ we can conclude that $F'(x) = f(x)$. This proof compares favorably with more standard proofs (Stewart, 2001, p. 385).

A historical vision is very useful also to show students how much has been gained since calculus was first developed at the end of the 17th century. The clepsydra problem is a particularly striking example (Rickey, 2002) since the Euclidean approach, developed by Mariotte in 1700, is quite complicated while the calculus approach is straightforward.

Most important of all, a historical perspective enlivens teaching and helps to avoid the pitfall of blurring the distinction between calculus and real analysis, providing an adequate framework to develop many calculus ideas from an intuitive understanding of limits, with occasional forays into more rigorous presentation. So, no $\varepsilon - \delta$ techniques are really needed in first-year calculus; they can wait until a course in real analysis. Let us remember that calculus developed successfully for almost 200 years (roughly 1670-1870) before Weierstrass and his pupils introduced the above-mentioned techniques. However, it is our responsibility as teachers to avoid any dangers inherent in a non $\varepsilon - \delta$ approach, letting students understand the difference between a plausible argument an a mathematical proof.

TECHNOLOGY

When we speak about technology we mean the use of graphing calculators that can do symbolic operations, the most prevalent technological tool in the calculus classroom. It is to be noted that during the last decade some interesting calculus projects using computers have developed into books (for instance Ben-Israel and Gilbert 2001). Whether the technological choice is graphing calculators or computers, it seems appropriate to set aside at least one hour per week for laboratory work.

First let me emphasize that in my opinion students need to learn how to differentiate and integrate simple expressions by hand, using for that purpose classical tools such as the chain rule, integration by parts, or integration by substitution. For more complicated expressions, especially those that appear in applications, technology can do the job for them once students are able to translate a word problem into coherent mathematics.

With regard to testing, we might divide calculus exams in two parts. In the first part students would not be allowed to use technology, while in the second part they would. Questions are to be constructed to reflect these circumstances. This is a debatable arrangement, not essential to the success of a calculus course.

Among the manifold issues raised by technology let me say something about programming, a subject that has not received the importance it deserves. I expect that my students will learn to build simple and short programs on their graphing calculators (TI-89 or TI-92 mostly), especially when a recursive process is involved, instead of copying them from a manual or book. Euler's method of approximation to the solution of an initial value problem $y' = F(x,y)$, $y(x_0) = y_0$ at a fixed point (close to x_0) is a typical example. The approximation of roots of polynomial equations through Newton's method is another example where the *For* command allows us to build a short program. Indeed, the recurrence formula $x_i = x_{i-1} - \dfrac{f(x_{i-1})}{f'(x_{i-1})}$ $i = 1,\dots,n$ obtained by looking at the problem graphically, is easily programmed (see appendix for an illustration of programs that students can build on their own, with some preliminary instruction from their teachers). By suitably choosing x_0 and n we can get a very accurate approximation to a root of a given function $y = f(x)$.

There are many other opportunities to use technology in an advantageous way. For instance, a simple program can be built if we wish that our calculator approximates $\int_a^b f$ through the trapezoidal (t), midpoint (m), and a version of Simpson's method with $s = \dfrac{t+2m}{3}$ (Hughes-Hallett and Gleason et al., 2002, p. 324). If students understand the underlying ideas behind the trapezoidal and midpoint methods they should be able to write for themselves the following simple program for any given function f defined on $[a,b]$ by an explicit formula:

```
Input "n", n
$\frac{b-a}{2n}\sum(f(a+(i-1)\frac{b-a}{n})+f(a+i\frac{b-a}{n}),i,1,n) \to t$
$\frac{b-a}{n}\sum(f(a+(2i-1)\frac{b-a}{2n}),i,1,n) \to m$
$\frac{t+2m}{3} \to s$
Display    t, m, s
```

It is to be noted that programs developed by students will work well only if they thoroughly grasp the underlying mathematical ideas. Thus, properly used, technology can become an incentive to mathematical learning.

CONCLUSIONS

The main goal of first-year calculus is to learn how to calculate and use derivatives and integrals, keeping in mind that the derivative at a point c of a function f is the slope of the tangent at $(c, f(c))$, while $\int_a^b f$ can be interpreted as an area, or sometimes as a volume, a force or some other physical concept. The five guidelines mentioned at the beginning of this paper help us focus on the main goal and the need to solve simple differential equations that arise in the study of concrete problems stemming mainly from physics and chemistry. How much emphasis will be put on each of the guidelines will depend on the intended audience. The key to success in teaching first-year calculus is, I believe, the ability to reach a balance between all competing tasks and obligations.

With some variations, the five guidelines can also serve as a framework when teaching other mathematics courses. For instance, when called upon to teach discrete mathematics, I try not to forget the need to show my students the process of discovery that often precedes the construction of a proof. For instance, the analysis of the quotient $\sum_{i=1}^{n} i^2 \Big/ \sum_{i=1}^{n} i$ allows us to conjecture that $\sum_{i=1}^{n} i^2 = \frac{n}{6}(n+1)(2n+1)$ (Polya, 1954, p. 108). Thereafter a proof by induction confirms our guess. This issue is related to our duty to look for explanatory proofs (Hanna, 1990).

REFERENCES

Baloglou, G. and Helfgott M. (2004). Finding equations of tangents to conics. *The AMATYC Review*. To appear.

Ben Israel, A. and Gilbert, R. (2002). *Computer-supported Calculus*. New York: Springer

Boas, R. (1971). Calculus as an experimental science. *Amer. Math. Monthly* 78, 6, 664-667.

Fauvel, J. and Gray, J. (1987). *The History of Mathematics, a Reader*. London: The Open University.

Frank, P. (1957). *Philosophy of Science*. Englewood Cliffs (N.J.): Prentice Hall.

Hahn, A.J. (1998). Basic Calculus, From Archimedes to Newton to its Role in Science. New York: Springer.

Hanna, G. (1990), Some Pedagogical Aspects of Proof. *Interchange*, 21, 1, 6-13.

Helfgott, H. and Helfgott, M. (2002). A noncalculus proof that Fermat's principle of least time implies the law of refraction, *Amer. J. of Physics*, 70(12), 1224-1225.

Hughes-Hallett, G., Gleason, A.M. et al. (2002), *Calculus* (third edition), New York: John Wiley & Sons.

Kline, M. (1967) . *Calculus: an intuitive and physical approach* (Vol.1). New York: John Wiley & Sons.

Lax, P. (1986). On the Teaching of Calculus, in *Toward a lean and lively calculus*: Conference/Workshop to develop alternative curriculum and teaching methods for calculus at the college level, Douglas R. (editor), MAA notes #6, 69-72.

Leinbach, L.C., Hundhausen, J.R. et al. (Editors) (1991) *The Laboratory Approach to Teaching Calculus*, MAA Notes #20.

Mosvold, R. (2003). Genesis principles in mathematics education, in Study the Masters: The Abel-Fauvel Conference, Bekken O. and Mosvold R. (editors), Nationellt Centrum for Matematikutbildning, NCM, Goteborgs Universitet, Goteborg.

Krantz, S.G. (1998). *How to teach mathematics*, 2nd ed., Providence: AMS.

Olson, D.A. (1997). First-year Students Love Calculus Proofs, *Primus*, VII, 2, 124-128.

Polya, G. (1954). *Induction and Analogy in Mathematics*. Princeton Univ. Press.

Rickey, F.V. (2002). *Abel-Fauvel Conference* (preprint). Kristiansand: Norway.

Simmons, G. (1985). *Calculus with Analytic Geometry*. New York: McGraw-Hill.

Schoenfeld, A. (Editor) (1997). *Student Assesment in Calculus*. MAA Notes #43.

Steen, L. (Editor) (1978), *Calculus for a new century: a pump, not a filter*. MAA notes #8

Toeplitz, O. (1963). *The Calculus: A Genetic Approach*. Chicago: The University of Chicago Press.

Tucker, T.W. (Editor) (1990). *Priming the Calculus Pump: Innovations and Resources,* MAA Notes # 17.

Tucker, T.W. (1999). On the role of proof in calculus courses. Contemporary Issues in Mathematics Education, *MSRI Publications*, 36, 31-35.

Young, A.L. (1996). Two familiar theorems: modified so their proofs are comprehensible to first semester calculus students. *Primus,* VI, 2, 107-115.

APPENDIX

Euler's Program

```
Input "$x_0$", x
Input "$y_0$", y
Input "n", n
Input "h", h
For  i, 1, n
y + hF(x,y)→ y
x + h → x
Endfor
Disp y
```

Newton's Program

```
Input "$x_0$", x
Input "n", n
For  i, 1, n
$x - f(x)/f'(x) \to x$
Disp x
Endfor
```