# INTRODUCING REAL NUMBERS: WHEN AND HOW?

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*Real numbers are widely used in high school and in university level calculus, but they are not properly defined. Even advanced calculus courses avoid getting into this tricky subject. The result is that students have only a vague idea about real numbers. To avoid misconceptions, and to allow a more meaningful study, it is essential to begin rigorously introducing real numbers at the early stages of the mathematical curriculum. We present here, in detail, a rigorous method, which is natural, enhances previous knowledge and leads to new concepts. The method is adaptable to both advanced high school programs, and to the beginning of tertiary level mathematics.*

## INTRODUCTION

While integers and rational numbers are carefully presented in school, real numbers, conceived as measures of continuous quantities, are taken almost for granted. Students at school are aware of the fact that there are numbers that are not rational, but the question of what "the collection of all numbers" is, is ignored. Freshman calculus courses are satisfied with illustrating real numbers as points on the *number line* (see for example Stewart, 1995). Though considered adequate for the course's needs, it actually amounts to a tautology. Advanced calculus courses also rush through this material. The formal presentation is usually postponed to topic courses in Real Analysis (see for example Rudin). This way only a few students are exposed to it. The result is that many students finish their university studies having only a vague idea of what real numbers really are. This can be demonstrated by raising questions such as:

- What is the meaning of **0.999…?** Is it less than 1? Equal to 1?
- How do we know that $\sqrt[3]{2}$, for example, is a real number?
- Your calculator's screen (limited to 9 digits) shows $1.35353535$ (alternatively $2.12345678$, or $3.14159265$) can you tell if your number is rational?
- How do we **add/multiply** two real numbers (like $\sqrt{2},\pi$)?
- How are real numbers **characterized**? How do we specify a particular real number? Can every real number be described (calculated)?
- How can we be sure that the Real Number System is **closed under** operations such as exponentiation, logarithms, etc?
- Historically, first examples of non-rational number resulted from **solving algebraic equations** such as $x^2-2=0$ (needed to measure the diagonal of the unit square). Is this is our motivation for extension? If so, how come not every solution of an algebraic equation (rational coefficients) yields a real number (e.g. $x^2+1=0$)? (And not every real number satisfies such an equation?)

Only after properly introducing real numbers, questions of the above type can lead to meaningful discussions. We will discuss here, in brief, some of those questions. Moreover, we propose that many well-recorded misconceptions concerning real numbers, result from the mere fact that real numbers are not properly introduced.

There are several rigorous methods of introducing real numbers. Most of them very formal and it is too difficult for beginners of tertiary level mathematics to grasp them. Moreover, the new ideas are too abstract for students to apply to questions of the above type. On the other hand, it is important that real numbers be properly introduced already at school, to address the possibility of gaps between students' concept image of real numbers and the concept definition, which does not exist at this stage (see Tall & Vinner, 1981). As for university level calculus courses, without first introducing properties of real numbers, one cannot prove the most fundamental theorems of calculus, and the emphasis shifts to computational aspects of the theory.

The solution to this puzzling situation**,** is to adopt a more natural way of introducing real numbers at the inter-phase between secondary/high school mathematics and tertiary mathematics (at the beginning of calculus courses.)

In this paper we are going to offer a systematic answer to both dilemmas of ***"how"*** and ***"when".*** We will describe here, in detail, a natural **geometric approach**, which is intuitive, relates well and enhances students' previous knowledge, and at the same time is mathematically sound.

## VARIOUS APPROACHES TO INTRODUCING REAL NUMBERS

***History in a capsule*** (adaptation from Wikipedia): The Egyptians had used fractions around 1700 BC; around 500 BC, the Greek mathematicians led by Pythagoras realized the need for irrational numbers. Negative numbers began to be accepted around 1600. The development of calculus around1700 used the entire set of real numbers without having defined them clearly. George Cantor can be considered the first to suggest a rigorous definition of real numbers in 1871.

Real numbers can be defined in two ways: axiomatic or constructive. We will only hint here at the most common rigorous definitions:

**Axiomatic approaches to defining real numbers:** The set of real numbers **R,** is defined abstractly by its properties: **R** is a complete (Archimedean) ordered field. Variations at this approach result from various definitions of the notion of **completeness.** Such presentation can be found in Tall (1977) or Spivak (1994).

**Rigorous definitional construction of real numbers:** The goal is to find an extension of the rational numbers that "contains its limits".

**Equivalence - classes:** Since we are motivated by limit arguments, we start by formalizing the idea of convergent sequences of rational numbers. We define **Cauchy sequences** of rational numbers. Real numbers are defined as **equivalence classes** of Cauchy sequences of rational numbers. We can define arithmetical operations and order, in terms of Cauchy sequences, and prove the required properties yielding a complete ordered field. See for example Rudin.

Note that, again, there are some variations to this method; e.g. it suffices to look at equivalence classes of bounded monotone sequences, etc.

**Dedekind cuts:** This is another abstract constructive method; see for example Landau or Hardy. Although the definition is geometrically motivated, and is based on a "natural idea", the definition is actually far too abstract to be efficient at the beginning of a calculus course.

## THE DECIMAL GEOMETRIC APPROACH - A RATIONALE

The approach suggested below was developed specifically for future secondary/high school mathematics teachers, training in Beit Berl Teachers' College. But it is appropriate for any tertiary level mathematics program. The rationale:

- Math teachers (actually any mathematician) should have an **intimate knowledge** of real numbers, as those are probably the most commonly used mathematical objects. Therefore, it is more appropriate to introduce real numbers *constructively rather than axiomatically.*
- Real numbers should be presented to future math teachers in a way that will
  enable them to **transfer** knowledge to their future students.
- The presentation should be **rigorous** mathematically (as vagueness leads to misconceptions), while at the same time **intuitive**, and relate to all types of possible (well documented) **misconceptions.**
- The necessary **axioms** should come as a natural byproduct of the construction process and *not be predetermined*.
- It is desirable to specify *in advance*, the **motivation** for extending an existing number system, and *act accordingly*. *The motivation can no longer be purely arithmetical*, as we expect real numbers to satisfy too many properties (closure under operations of exponents, logarithms, etc). Completeness as a motivational property is too obscure for students at this stage and leads to definitions that are too abstract for them.
- An introduction to real numbers at this early stage, should not start with "big ideas", but should eventually lead to such ideas (convergent sequences / series).
- It is desirable that the method be illustrated using computerized programs.
- The number line is a good starting point. It is easy to define **geometric objectives**: a number system that will enable us to measure distances in the plane, e.g. diagonals of rectangles, etc.

## THE DECIMAL GEOMETRIC APPROACH, AN OVERVIEW

**A geometric motivation**: The necessity to extend the rational number system stems from the realization that we cannot measure simple geometric entities, such as the diagonal of the unit square, by rational numbers. Instead of translating such problems into arithmetical terms (using Pythagoras' theorem), we stick to geometry. We try to construct a **"(virtual) ruler"** with which we can measure the distance between any two points on a plane.

Noticing that there are "holes" in the "rational numbers ruler" (see Fig. 1), we look for a method of "filling the gaps", so that each point on the ruler will have a corresponding "number" that measures its distance from the origin.

Note that we use decimal notation for numbers. We could easily have used other bases (like base 2), but base 10 is the most common.

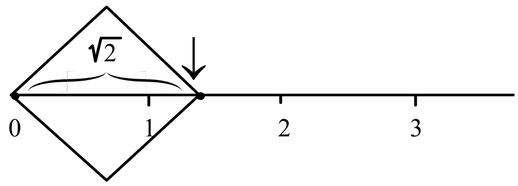


Fig 1: a gap in the rational ruler

**Step I:** To extend the set of rational numbers, we first have to produce an *"extendable presentation"*; this motivates us to express rational numbers in their **decimal expansions** (obtained as outcomes of the long division algorithm).

Since we may get infinite expansions (or even finite, but too long for practical purposes), to each such expansion we adjoin its sequence of **truncations**. We notice that each decimal expansion can be **approximated** by one of those truncations, with **any desired degree of accuracy.**

**New ideas**: the notion of **convergence of the sequence of truncation to the infinite expansion arises naturally here.**

**Step II**: A **geometric interpretation,** on the rational ruler, to the decimal expansions we obtained algebraically: We define a **dynamic process** of finer and finer **decimal partitions** of unit intervals (intervals of length 1) on the ruler. The expansion is interpreted in terms of partition intervals.

Noticing that our method is extendable to all points on the ruler, we associate with **any point** on our ruler a unique decimal expansion, obtained via an **infinite "accumulating sequence" of finite expansions**.

**New ideas:** Convergence of a sequence of rational numbers can be redefined in dynamic geometric terms using decimal partitions of intervals.

**Step 3:** Real numbers are now defined abstractly as **all possible decimal expansions.** Obviously all rational numbers are real (but not vice versa). To each real number we adjoin its sequence of **truncations** (an infinite accumulating sequence). Using such sequences it is easy to extend definitions of **order and arithmetical operation** to real numbers, and to prove **field and order properties**.

**Completeness** is also easily proven using truncations.

**Step 4**: Combining the above steps: All decimal expansions on the ruler are actually real numbers (measuring distances from the origin). Using the Hilbert completeness axiom of geometry, it is easy to establish a one-to-one correspondence between points on the line and real numbers (not ending with infinite recurrence of 9).

**Step 5**: Fruitful discussions. Questions like those posed in the introduction, are a good basis for in-class discussions that enhance the understanding of the new ideas.

In the rest of this presentation we will discuss in more detail the "how" and the "when" dilemmas. We will also present (pre-calculus) discussions of some of the questions on our list and offer additional discussions of the above type.

## DECIMAL EXPANSIONS OF RATIONAL NUMBERS

We consider here only rational numbers between 0 and 1 (the presentation easily extends to the entire ruler).

***Decimal expansion of a rational number***: For a positive rational number $r = m/l$ we **define** its (unique) decimal expansion as the outcome of a long division algorithm of $m$ by $l$. The expansion can easily be proved to be one of two types:

- **Finite,** $r = 0.k_1k_2\cdots k_n$: If $r$ can be represented as $r = m/10^n$; or otherwise
- **Periodic:** It ends with infinite recurrence of a nonzero block of digits.

(One could add zeros at the end of a finite expansion and consider it a periodic expansion, but we will stick to the term finite expansion, because it plays an important role in the forthcoming theory.)

We redefine **order** (lexicographic) and **arithmetical operations** in terms of expansions. The new definitions are consistent with previous definitions (for fractions), and thus we get an ordered field.

***The sequence of truncations***: To understand the **meaning** of an infinite expansion $r = 0.k_1k_2k_3\cdots$, we will have to look at its sequence of truncations: $r_1 = 0.k_1,\ r_2 = 0.k_1k_2,\ r_3 = 0.k_1k_2k_3, \cdots$. Note that $r_1,\ r_2,\ r_3, \cdots$ all have finite expansions, also $r_1 \le r_2 \le r_3 \le \cdots$ (monotonicity).

**Example:** The sequence of truncations of $1/3 = 0.333\cdots$ is $0.3 \le 0.33 \le 0.333 \le \cdots$. Let $r_n = 0.\underbrace{3\cdots3}_{n}$. Clearly $|r_n - 0.33\cdots| = 0.\underbrace{0\cdots0}_{n}33\cdots < 0.\underbrace{0\cdots0}_{n}4 = 4/10^{n+1} < 1/10^n$.

In other words, the term $r_n$ of the truncation sequence, approximates the infinite expansion $0.333\cdots$ with degree of accuracy of $1/10^n$.

E.g. $r_3 = 0.333$ approximates 0.333…with degree of accuracy of 0.001, etc.

***Approximating an infinite expansion***: Given any decimal expansion of a rational number $r$, the term $r_n$ of its truncation sequence satisfies $|r_n - r| < 1/10^n$.

Note that by choosing $n$ large enough, $1/10^n$ can be made as small as we please. Hence, terms $r_1,\ r_2,\ r_3, \cdots$ of the adjoined sequence of truncations, serve as natural **approximations** to $r$, with any desired degree of accuracy.

We are now at a very good position to introduce important subject of **convergence:**

***Convergence of the sequence of truncations of a rational number $r$ :*** For **any** fixed $d = 1/10^K$ (as small as we wish), truncating the expansion of $r$ after the $K$'th digit, we get $|r_K - r| < 1/10^K = d$.

Since our sequence is monotone, we actually proved:

$$\cdots \le |r_{K+3}-r| \le |r_{K+2}-r| \le |r_{K+1}-r| \le |r_K - r| < d .$$

In other words, for all $n \ge K$, $|r_n - r| \le |r_K - r| < d$.

Terminology: we say that the sequence of truncations $r_1, r_2, r_3, \cdots$ of $r$, converges to $r$.

***A new concept - Convergent sequences of rational numbers***: The above definition can easily be generalized to **any** sequence of rational numbers:

*A sequence $r_1, r_2, r_3, \cdots$ of rational numbers converges to a rational number $r$, if:*

*For any fixed $d$, there exists a number $K$, such that for all $n \ge K$, $|r_n - r| < d$.*

(It seems preferable to first discuss convergence of monotone sequences, in which case the definition is simpler: For any fixed $d$, there exists a term $r_K$, of the sequence, that satisfies $|r_K - r| < d$.)

Relevant examples here are:

**Example 1 (geometric sequence)**: For any $0 < q < 1$ rational, the geometric sequence $1, q, q^2, q^3, \cdots$ converges to 0.

**Example 2 (infinite geometric series)**: For any $a$, $0 < q < 1$ rationals, consider the finite geometric sums $s_n = a + aq + aq^2 + \cdots + aq^n = a(1-q^{n+1})/(1-q)$.

The sequence $s_1, s_2, s_3, \cdots$ converges to $a/(1-q)$. We say that the infinite geometric series converges, and we write $a + aq + aq^2 + aq^3 \cdots = a/(1-q)$.

***Representation theorem:*** All terms of the sequence of truncations of $r = 0.k_1k_2k_3\cdots$,

can be expressed as finite sums: $r_n = 0.k_1 \cdots k_n = \frac{k_1}{10} + \cdots + \frac{k_n}{10^n}$.

Since the sequence of finite sums $r_1, r_2, r_3, \cdots$ converges to $r$, we can represent $r$ as "infinite decimal series": $r = 0.k_1k_2k_3 \cdots = \frac{k_1}{10} + \frac{k_2}{10^2} + \frac{k_3}{10^3} + \cdots$. (*)

****A new concept- Convergent series:*** It is possible to introduce here the general topic of convergent series. We already have two good examples: convergent geometric series, and convergent "infinite decimal series" (see (*)).

**Discussion**: Note that **any** "pure" periodic decimal expansion (only 0's before the recurrent block of digits) is actually a convergent geometric series.

**Example 1:** $0.333\cdots = \frac{3}{10} + \frac{3}{10^2} + \frac{3}{10^3} + \cdots$. The right-hand side is a convergent geometric series $(a = 3/10, q = 1/10)$; its sum is $(3/10)/(1-1/10) = 1/3$.

**Example 2:** $0.0121212\cdots = \frac{12}{1000} + \frac{12}{10^5} + \frac{12}{10^7} + \cdots = \frac{12}{990}$ $(a = 12/1000, q = 1/100)$.

***Discussion- characterization theorem:*** Every finite expansion corresponds to a (unique) rational number. Does the same hold for every periodic expansion?

We discover that every periodic expansion **not** ending with infinite recurrence of 9, results from the long division algorithm, and is therefore an expansion of a (unique) rational number. On the other hand, an expansion ending with infinite recurrence of

9**, cannot** result from the long division algorithm. In other words there is no rational number having this type of decimal expansion.

***Discussion- periodic expansion ending with infinite recurrence of 9:*** So far, 0.99..., for example, has no meaning! On the other hand, such an expression cannot just be ignored, because it will spoil the characterization of rational numbers by their decimal expansion. Moreover, since $r = 0.333\cdots$ is well defined, $3 \times r$ should also be well defined (and consistent with multiplication properties). Our goal now should be to define 0.999… in a way, which is **consistent with** all previous results.

Note that the sequence of truncations $r_1, r_2, r_3, \cdots$ of $0.999\ldots$ is well defined: $r_n = 0.\underbrace{9\cdots9}_{n} = \frac{9}{10} + \cdots + \frac{9}{10^n}$, i.e. $r_n$ is a finite geometric sum. The sequence $r_1, r_2, r_3, \cdots$ converges to the corresponding infinite geometric series. Hence **the only consistent definition** of $0.999\ldots$ is: $0.999\cdots = \frac{9}{10} + \frac{9}{10^2} + \frac{9}{10^3} + \cdots$. Using the formula for the sum of a convergent geometric series established earlier, we are led to the conclusion:

**Corollary 1**: 0.999…=1.

This result is consistent with the fact that $3 \times 0.333\cdots = 0.999\cdots$, while $3 \times 1/3 = 1$, etc. Other expansions ending with repetitions of 9 may be handled in a similar way.

**Corollary 2**: The representation theorem (*) holds for **any** periodic expansion.

**Corollary 3:** rational numbers are **characterized** by the property that their decimal expansions are either finite or periodic.

***On the question of "When":*** Computers and calculators present rational numbers in their truncated decimal expansion. So, school students should have an understanding of decimal expansions and of their **sequences of truncations**. Note that in practice we cannot present, nor use, all digits of an infinite expansion (or even of finite but long expansions). Thus it is important for school students to know that truncations serve as **approximations** to $r$ in computational problems, and that they offer a possibility of control on the level of accuracy. Examples should emphasize the **dynamical nature** of the truncations. Customized **computer programs** are very helpful here (see Barthel 2004). Note that the use of calculators here may lead to misconceptions, confusing a number with its partial expansion as appears on the calculator's screen (see Pinto & Tall 1996).

Infinite **geometric series** should also be discussed in school, at this point, as it comes out naturally in the discussions and in examples.

Discussion of **convergence** of the sequence of truncations can be postponed to the beginning of a calculus course. Convergence here is easily established because our sequences have extremely nice properties; it is simple to generalize the new idea **later in the course** (first to monotone sequences and then to any sequence of rational numbers). Thus we have gained a very natural introduction to the general concept of convergence, which is a central concept in calculus. Again, it is easy to visualize convergence of the sequence of truncations using computerized programs (see module developed by Barthel 2004).

## BUILDING A (VIRTUAL) RULER - A GEOMETRIC ALGORITHM.

We start with the "rational ruler"- all non-negative rational numbers are expressed on it in their decimal expansion. We are now ready "to fill the gaps" on the ruler.
Again we will discuss here only points between 0 and 1.

***Decimal divisions of the unit interval:*** We propose a geometric dynamic process of finer and finer partitions of the unit interval: The unit interval is divided into 10 equi-length intervals, referred to by their numbers 0,1,…,9 (left to right). Each such interval is partitioned again into 10 subintervals, etc.

The lengths of the intervals along the process are $1/10, 1/10^2, \cdots, 1/10^n, \cdots$. The set of partition points are all the rational numbers on the ruler, having finite expansions. To each **non-partition point** $x$, we associate an infinite sequence of finite expansions defined as follows:

At step 1, $x$ is an interior point of one of 10 intervals of length 1/10. The interval number is denoted $k_1$. The left endpoint of this "enclosing interval" is $r_1 = 0.k_1$.

At step 2, $x$ is an interior point of one of 10 **subintervals** of length 1/100. Let $k_2$ be the reference number of that enclosing interval, and $r_2 = 0.k_1k_2$ its left endpoint. Continuing this way, with each non-partition point $x$, we associate an infinite non-decreasing bounded sequence of rational numbers: $r_1 = 0.k_1, r_2 = 0.k_1k_2, r_3 = 0.k_1k_2k_3, \cdots$.

Note, that the terms of this sequence are expansions of a very special type: $r_1, r_2, r_3, \cdots$ is an **accumulating sequence,** where $r_{n+1}$ is obtained from $r_n$ by cumulating one digit - $k_{n+1}$, at the end of the expansion ($k_{n+1}$ is the reference number of the enclosing interval at step $n+1$ of the decimal process of partitions).

***Decimal expansions of non-partition points***: With each non-partition point $x$, we associated an accumulating sequence $r_1 = 0.k_1,\ r_2 = 0.k_1k_2,\ r_3 = 0.k_1k_2k_3, \cdots$.

We can now define a unique infinite decimal expansion: $x = 0.k_1k_2k_3\cdots$, and write this expansion on our ruler, at the point $x$.

***Special case***: Note that if $x$ is **rational,** this expansion agrees with the existing expansion as already appears on our ruler. Furthermore, its accumulating sequence $r_1, r_2, r_3, \cdots$ is its sequence of truncations, which, as we proved, converges to $x$.

***Summary:*** We have accomplished our first goal: we extended the notation on our ruler, thus filling in all the "gaps". We now have a continuum. *We are not done yet* though, because for non-rational points, the expansions still have no meaning!

***On the question of "When":*** We consider it important to discuss, already at school, the geometric interpretation of decimal expansions of rational numbers (both of partition and non-partition points). Describing the outcome of the algebraic algorithm of long division in terms of a process of decimal partitions of intervals enhances deeper understanding of the dynamic nature of infinite expansions of rational numbers, and is easily illustrated by computerized programs.

Pre-calculus presentations can end here: we managed to associate with each point $x$ on the ruler a unique decimal expansion in a way that extends decimal expansions of rational numbers; thus obtaining the collection of real numbers. Moreover its sequence of truncations serves as approximations of the distance from the origin.

## REAL NUMBERS – FORMAL DEFINITION AND PROPERTIES.

**Real numbers are defined abstractly as all decimal expansions of the type** $x = m + 0.k_1k_2\cdots$ **(**$m$ **integer,** $k_1, k_2, k_3, \cdots$ **digits)** (see also Gowers 2003):

Clearly every rational number is also a real number (but not vice versa). We can easily extend the definitions of order, convergence and arithmetical operations from rational numbers (given in their decimal expansions) to real numbers. We first define:

***The sequence of truncations***: With each real number, we associate its infinite "sequence of truncations" $r_1(x), r_2(x), r_3(x), \cdots$ (all rational numbers) defined by:

$$\overbrace{\overbrace{\overbrace{m + 0.k_1}^{r_1 = r_1(x)}\, k_2}^{r_2 = r_2(x)} \cdots k_n}^{r_n = r_n(x)} \cdots .$$

We use these truncations to extend the definition of **order** (lexicographic). To be able to extend the definition of convergence, we first define convergence of sequences of rational numbers $x_1, x_2, x_3, \cdots$ in terms of truncations:

A sequence of rational numbers $x_1, x_2, x_3, \cdots$ , converges to a rational number $x$, if for **every** $n$, there exists an element, $x_K$, of the sequence, such that $r_n(x_{K+1}),\ r_n(x_{K+2}),\ r_n(x_{K+3}), \cdots$ (the truncations at $n$) are all **equal to** $r_n(x)$.

We now have an "extendable definition":

***General definition of convergence***: An infinite sequence of real numbers $x_1, x_2, x_3, \cdots$ converges to a real number $x$, if for **every** $n$ there exists an element, $x_K$, of the sequence, such that $r_n(x_{K+1}),\ r_n(x_{K+2}),\ r_n(x_{K+3}), \cdots$ (the truncations at $n$) are all equal to $r_n(x)$. $x$ is called a **limit** of $x_1, x_2, x_3, \cdots$.

***Convergence of the sequence of truncations:*** Using the above definition, we can easily extend a previous result:

The sequence of truncations $r_1(x), r_2(x), r_3(x), \cdots$ of a real number $x = m + 0.k_1k_2\cdots$ converges to $x$.

***Completeness****:* We now come to the main feature of the set real numbers, that they contain their limits. We can easily prove at this stage that:

*Every bounded non-decreasing sequence of real numbers converges to a real number $x$.*

As we know, the system of rational numbers does not satisfy this property:

**Example:** $x = 0.123\cdots9101112\cdots$ is a real number that is not rational (it has non-periodic expansion). Therefore its sequence of truncations is a convergent sequence of rational numbers, which does not converge to a rational number.

***Arithmetical operations on real numbers:*** Using completeness arguments it is now easy to define summation and multiplication of real numbers:
If $x, y$ are real numbers, then $x + y$ is defined as the limit of the bounded non-decreasing sequence $r_1(x)+r_1(y),\ r_2(x)+r_2(y),\ r_3(x)+r_3(y),\cdots$.
The product $x \cdot y$ is defined analogously.
It is easy to extend familiar arithmetical properties from rational numbers to real numbers, deducing:

*Real numbers constitute an* ***ordered field*** *under summation and multiplication.*

***More operations defined:*** Using completeness arguments, it will be possible (later in a calculus course) to show that real numbers are closed under operations such as exponentiation, etc.

**THE UNIFIED APPROACH; DISCUSSIONS**

Unifying our geometric approach with the above abstract approach, we note that all the expansions on the extended decimal ruler are real numbers.
We define **distance** between any two points $x, y$ on the ruler by: $d(x, y) = |x - y|$ (all standard properties of distance are easily verified). We conclude that the real number written at the point $x$ on our ruler is equal to its distance from the origin. Moreover, its sequence of truncations serves as approximations to the distance with any desired degree of accuracy.

***Do all non-negative real numbers appear on the ruler?*** Using the sequence of decimal partitions of the unit interval, to each real number $x$ having infinite expansion $0.k_1k_2\cdots$, we can associate an infinite sequence of **closed nested intervals** of lengths $1/10, 1/10^2, \cdots, 1/10^n, \cdots$ (interval $k_1$ of step 1, interval $k_2$ of step 2 etc). It is natural to expect that all those intervals have a common point, to be denoted by $x$. In this context the necessary Hilbert completeness axiom of the line arises in a very natural way. It can easily be shown that $x$ is unique. Moreover, if $0.k_1k_2\cdots$ does not end with infinite recurrence of 9, $0.k_1k_2\cdots$ is indeed the expansion written on the ruler at the point $x$ (its distance from the origin).

***Real numbers ending with infinite recurrence of 9:*** Trying to track a point on the ruler (using the above method) whose distance from the origin is given by 0.0999…, we end up with the partition point 0.1. A similar phenomenon happens with every real number ending with infinite repetitions of 9.
**To achieve consistency** with all previous results, **we are again led to define 0.0999…=0.1, 0. 999…=1, etc**. From each such pair, only the finite expansion will appear on our decimal ruler (always by a partition point). In other words, decimal expansions ending with infinite recurrence of 9, are real numbers (in fact rational numbers) but they do not appear, as is, on our extended ruler. To each such

expansion, there exists a finite expansion, which (by definition) is equal to it and appears on the ruler.

***Convergence of sequence of points on the real line redefined in geometric terms***: Using the sequence of decimal partitions of unit intervals on the line, we can now Redefine convergence in dynamic geometric terms in a way that is very intuitive.

*An infinite sequence of points on the real line $x_1, x_2, x_3, \cdots$ converges to a point $x$, if: for **every** $n$, there exists a point, $x_K$, in the sequence, with the property that all subsequent points $x_{K+1},\ x_{K+2},\ x_{K+3}, \cdots$, belong **at step** $n$ of the partition process, **to the same partition interval** as $x$.*

***Examples of non-rational real numbers - first method***: We know that $\sqrt{2}$, $\sqrt{8}$, $\sqrt{18}$

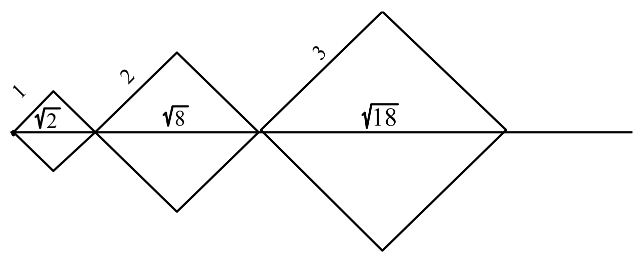


Fig 2: $\sqrt{2}, \sqrt{8}, \sqrt{18}$ are real numbers

etc, are not rational numbers. It is easy to see that they are all real numbers. Indeed, for any number in this list, one can easily draw two points on the ruler, the distance between which is the given number (see graph above):

**Discussion:** The decimal expansions of these numbers are non-periodic. *There is no way we can actually specify all the digits in their infinite expansion*! But using geometric arguments we can give better and better approximations (obtained via the sequence of enclosing intervals).

**Example**: Using a calculator we can easily verify that if $x$ satisfies the equation $x^2 = 2$, then $1.4 < x < 1.5$, and $1.41 < x < 1.42$, $1.414 < x < 1.415$,…(all those are partitions intervals). In this way, we can compute more and more terms of the sequence of truncations $1.4, 1.41, 1.414, 1.4142, 1.41421, \cdots$

E.g. $r_{10} = 1.4142135623$, approximates $\sqrt{2}$ with accuracy of $0.0000000001$.

***Examples of non-rational real numbers - second method***: To prove that $\sqrt[3]{2}$, $\sqrt[5]{3}$ etc, are real numbers, we can show a way of computing their accumulating sequences.

***Example:** Is there a real number solution to the equation $x^3 = 2$?

Using a calculator again, we can easily verify that if $x$ satisfies the equation $x^3 = 2$, then $1.2 < x < 1.3$, $1.25 < x < 1.26$, $1.259 < x < 1.260$, etc. We can compute this way more and more terms of the truncation sequence $r_1 = 1.2,\ r_2 = 1.25,\ r_3 = 1.259, r_4 = 1.2599, \cdots$. The resulting sequence $r_1, r_2, r_3, \cdots$ is a bounded non-decreasing sequence of real numbers. Using completeness we can deduce that it converges to some real number $x$. It remains to show that $x^3 = 2$. This is achieved by showing $r_1^3, r_2^3, r_3^3, \cdots$ converges to 2.

***Examples of non-rational real numbers - third method***: Is $\pi$, say, a real number? It suffices to come up with a bounded non-decreasing sequence of real numbers converging to $\pi$. Later on in the course, we will be able to present a non-decreasing sequence of rational numbers converging to $\pi$.

***On the question of "When":*** The one-to-one correspondence discussion can be postponed to a calculus course, but it may be beneficial to discuss (using geometric terms) the case of 0.999…. already in school.

Establishing the fact that $\sqrt{2}, \sqrt{8}, \ldots$ are real number can easily be done in school. Using calculators we can easily compute more and more terms of the sequence of truncations of $\sqrt{2}$, say. The question whether $\sqrt[3]{2}$, say, is a real number better be postponed to the beginning of a calculus course. Generalization to $r^s$ ($r > 0, s$ rationals) will be discussed only later in the course.

## EPILOGUE

Returning to our teachers college program, we devised a special first year transition (pre-calculus) course, devoted entirely to the subject of number systems. By the end of this course, students are mature enough for interesting discussions like: Isn't it an overkill to define a system of numbers many of which we cannot even describe (calculate). Or, if we will never be able to give the entire infinite expansion of say, $\pi$, or $\sqrt{2}$, and end up dealing with rational numbers, what did we gain?

There are also interesting practical questions, like: How do we actually add two real numbers? This question leads to interesting observations about the sum of the two associated sequences of truncations. A good way to investigate properties of real numbers in decimal expansions is with the aid of computerized programs. The graphing and computing abilities of CAS software like Mathematica, can illustrate our above presentation by offering visualization of the dynamic geometric process and by specific examples of computations. An interactive such module which students can use to experiment with a concrete manner is described in Barthel (2004).

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