A PORTUGUESE STUDY ON LEARNING CONCEPTS AND PROOFS:

MULTIVARIABLE CALCULUS AND MATHEMATICA®

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Infinitesimal Calculus in our Universities is traditionally presented in the so-called form of "pencil and paper" teaching and learning strategies. Nevertheless one is dealing with subtle concepts that have, at present as well as historically, presented difficulties of understanding to students who arrive at our Universities. What we are presenting here is a summary of some of our teaching experiments involving 1st year Informatics' students who, through their 2nd semester at the University were exposed, in their Multivariable Calculus course, to working in groups with a CAS (Computer Algebra System). The results of these changes were evident straight away, in the form of an improvement in the students' attitudes towards the subject and, at the end of the semester, in the form of a considerable improvement in the pass rate for the course.

Key Words: Calculus and MATHEMATICA®, University Teaching Methods.

INTRODUCTION

Learning Infinitesimal Calculus is, at present as well as in the past, evaluated in terms of the ability to manipulate a variety of processes, frequently artificial ones which include intuition as well as algorithms (formulas) together with formalization of basic concepts such as domains, limits, continuity, derivatives and primitives of functions.

In Portugal and through traditional "paper and pencil" teaching we have experienced, with 1st year Science and Engineering students at our Universities, the same kind of conceptual difficulties that are widely reported in studies worldwide [Tall91]. But we are also involved within an educational system that allows a student to repeat, whatever the justification might be, a course until a 50% evaluation mark is achieved in a "paper and pencil" final examination; the result being that in each academic year, in each Calculus course, the lecturer is teaching to a heterogeneous group of students who span from 1st registration students to 6th, 7th or even 13th year registration students who are attempting to learn (pass) a 1st year course. The result of this situation is that a class size of 450 students is "normal" for a lecturer to have to run in a semester, and a pass rate of 30% is "normal" in a 1st year Calculus course.

During the Spring semester of 2002, two of the authors taught a 1st year university course on Multivariable Calculus to Engineering students in Systems and Informatics. In the light of the previously cited international and national studies, we decided to <u>develop a teaching strategy which aimed to increase the students' pass rate on the course</u> without decreasing the level of concept acquisition (mathematical knowledge). We also wanted to specifically deal with the extra difficulty of lecturing Calculus to such a heterogeneous group of students.

The common denominator of these students was, as one might easily expect, neither age nor previous knowledge of the subject but rather a specific motivation for using New Technologies (whatever the previously acquired experience on the subject).

Within the study the authors adopted a qualitative approach using different techniques for gathering data, particularly questionnaires, class observations and document analysis. We did not change/decrease the size of the traditional classes (200 in lectures and 65 in tutorials) but we <u>adjusted the traditional working pattern of each tutorial group</u> by using the allocated 3 hours a week partitioned into:

i) 1h45m - "students' working on their own" - the class was divided into groups of 6 (on average) and each group was given precise directions from the lecturer for solving specific parts of the exercises/problems that had been set;

ii) 1h - "students' sharing solutions" – a representative of each group presented orally the group's solutions to the other groups.

iii) 15m-“lecturer summarizing the fundamental ideas of the session” – linking them to previous knowledge and/or subsequent conceptual tasks.

In addition, we complemented most of the concepts/proofs occurring using a Computer Algebra System (CAS), namely *Mathematica*®. The students were, therefore, motivated to draw from such working conditions the analysis, the discussions and the opportunity to deepen their knowledge of Multivariable Calculus. The basic idea was to enable our students to achieve high levels of logical-analytical reasoning by visually supporting, directly or indirectly, the concepts and the proofs with graphics presented through *Mathematica*®. Our consciously planned goal was establishing communication channels among pictures (graphics), numbers, symbols and words (written and spoken) related to the Infinitesimal Calculus of Multivariable Functions.

PREPARING THE EXPERIMENT

We had already been involved in teaching traditional Calculus courses for Science and Engineering degrees (both in Portugal and in England) and it was not difficult to observe students dealing with graphics, formulas, specific symbols and adequate terminology in a way that other authors have reported [Hirst02] and that would hardly be compatible with the acquisition of the fundamental Calculus concepts. One of the things that we tested before undertaking the experiment was achieved by means of a questionnaire on the students’ opinions about Calculus courses. This revealed that

70 % of students (in a total of 200) agreed on the importance of the course; 50 % agreed that content was relevant, independently of the degree to which the course was addressed; 85 % thought that the course should be kept compulsory, independently of the curricular reforms taking place at the University.

There is now a wide consensus that the available mathematical software, and specifically the visualization capacities offered by many of these packages, can significantly contribute to the learning of Mathematics [Zorn99]. The lecturers involved in teaching the Calculus course where the experiment was conducted, had also decided to take advantage of these graphical capabilities. However this teaching experience was not planned to study/deal with the, also widely documented, aspects of linking the graphical capacities of the New Technologies to the confirmation of mathematical intuitions or even in the formulation of conjectures [Simpson98]. The present experiment intended, in the first place, to explore the role of the visual representations that were offered by a package such as Mathematica®, in creating conceptual doubts in the students; these same doubts might eventually lead the students to verification/demonstration of the results that were being presented to them by their own mathematical/informatics investigations, by their colleagues' challenges or by their lecturer's comments/questions. We aimed therefore to attain students' true understanding of basic Calculus concepts, through "convincing arguments" used to communicate both among colleagues and with the lecturer rather than through either mere imitation or memorized information.

SOME OF THE CHOSEN EXAMPLES

1. Domains (of two variable functions)

We took a spherical surface, as the graph of two real functions of two real variables. We made use of the following instructions:

p1=Plot3D[Sqrt[1-x^2-y^2],{x,-1,1},{y,-1,1}];

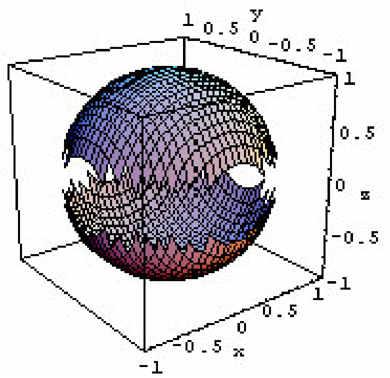
p2=Plot3D[-Sqrt[1-x^2-y^2],{x,-1,1},{y,-1,1}];

Show[{p1,p2}];

Using cartesian coordinates, within the simplest instruction for these graphs, **Plot3D**, the function defined in a circle whose radius is 1, using a square of side 2, namely **{x,-1,1},{y,-1,1}**, leads to square roots of negative numbers; and the program informs us that it was not possible to obtain a list of three real numbers, in order to draw the graphic.

The result of such conditions is a lack of points close to the border of the domain. The final picture, as following, was presented to the students for them to comment on it:

$$z=\sqrt{1-x^2-y^2} \text{ and } z=-\sqrt{1-x^2-y^2}$$

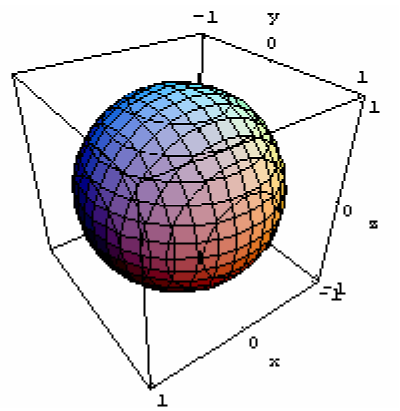


The typical students' answer was as follows:

> *Since the surface was algebraically divided into two, the points where* $z=0$ *(* $z=\sqrt{1-x^2-y^2}$ *and* $z=-\sqrt{1-x^2-y^2}$ *), are taken twice (belong to both hemispheres) and therefore the computer gets confused and ignores such cases, the program can not "decide" with which surface they are to be drawn, so they are suppressed.*

2. Level Surfaces

This same problem was subsequently used for dealing with functions of 3 variables; this time the surface was to be viewed as a level surface of the function $f(x,y,z)=x^2+y^2+z^2$ with constant equal to 1; the following *Mathematica*® instructions were used:



<<Graphics'ContourPlot3D'

ContourPlot3D[x^2+ y^2+z^2,{x,-1,1},{y,-1,1},

{z,-1,1},Contours->{1.}];

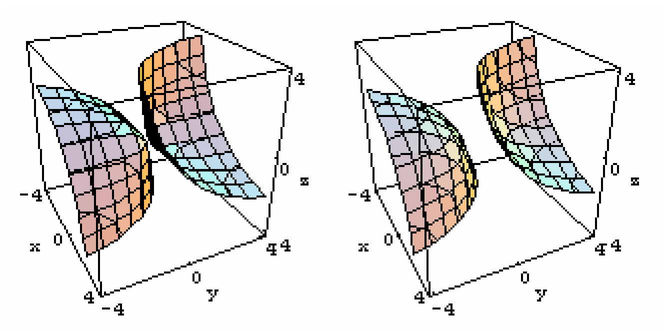
and we obtained the following picture:

The discussion on the role of the square root on the previous example proved crucial for the students to understand what had really happened to the first graphic of the sphere. Domains were subsequently treated by these students with a proficiency that we had never seen students using before, i. e., geometry and algebra (visualization and technique) were finally hand in hand in students acquisition of knowledge.

From level surfaces, still necessary in the context of, for example, tangent planes, we guided the students to encounter some surfaces with what seems to be the "right look". We pushed them into commenting things such as:

```
<<"Graphics‘ContourPlot3D‘"

p1=ContourPlot3D[x y+y z+x z,{x,- 4,4},{y,-4,4},{z,-4,4}, Contours->{0.}];

p2=ContourPlot3D[x y+y z+x z,{x,-4,4},{y,-4,4},{z,-4,4}, Contours->{2.}];
```

$x\,y + x\,z + y\,z = 0$ and $x\,y + x\,z + y\,z = 2$



by asking the students:

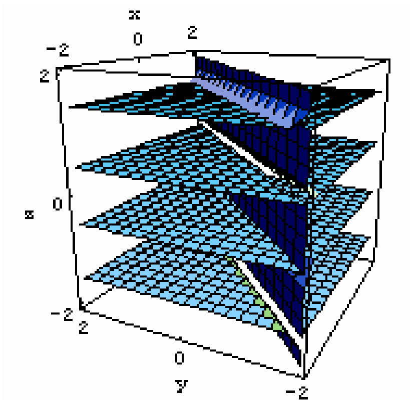
Does it look right? Why?

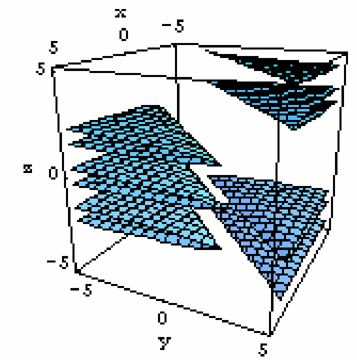
It did not take long before students were all agreeing on this case having the right look (in accordance with the algebra). But in other cases the solution did not come as quickly, namely, other problems raised some doubts. By instructing *Mathematica*® with the following instructions:

```
<<"Graphics‘ContourPlot3D‘"

ContourPlot3D[(x-y) Cos[Pi*z], {x,-2,2},{y,-2,2},{z,-2,2},Contours->{0.}];

ContourPlot3D[(x-y) Cos[Pi*z], {x,-5,5},{y,-5,5},{z,-5,5},Contours->{2.}];
```

We have obtained the following graphics:

$(x - y)\mathrm{Cos}[\pi z] = 0$ and $(x - y)\mathrm{Cos}[\pi z] = 2$





In the students' own words:

> *In this case, and for a value of the constant equal to 0, it would have been quite simple to take an algebraic approach and forget all about pictures…*

The students therefore had the opportunity to realize possible misconceptions arising from graphics, as well as to report on their thoughts. This proved very important for the lecturers to try to follow their learning directly and to rearrange the teaching in accordance with a step by step process in developing their concept learning.

But, in this specific case, we did not want them to forget/ignore all the pictorial information. In fact, we wanted them to be more careful about taking for granted the pictures that the computer was offering to them, i. e., to be aware of potentially mistaken ideas drawn from looking at the pictures.

Once again by making our students discuss such examples we noticed substantial improvements in the way they started looking at trigonometric functions. Before this experience took place, such type of functions would necessarily provoke on the students some kind of uneasy attitude; whereas this time we have seen them dealing with trigonometric (logarithms, exponentials) with no noticeable discomfort.

3. Maxima and Minima

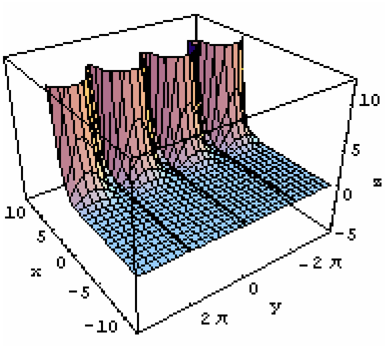
We approached the following example ($z = e^x (1 - Cos[y])$) using a graphic with a domain big enough to allow us to have a general notion of its appearance, which the students were required to confirm (by means of an algebraic/analytical proof) or to dismiss the visual impression.

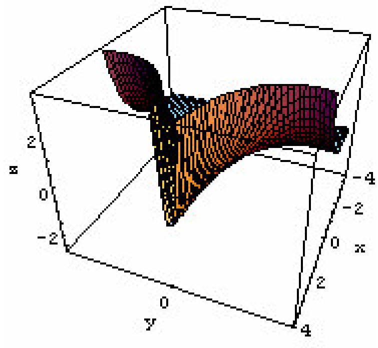
Later on, the students required an adequate zoom on one of the minimums (in particular, for $y = 0$) with an appropriate rotational view.

They could explore, once more, the lack of visual information this time on the existence of minima.

Plot3D[Exp[x]*(1-Cos[y]), {x,-11,11},{y,-11,11}];

Plot3D[Exp[x]*(1-Cos[y]),{x,-4.5,4.5},{y,-4.5,4.5}];

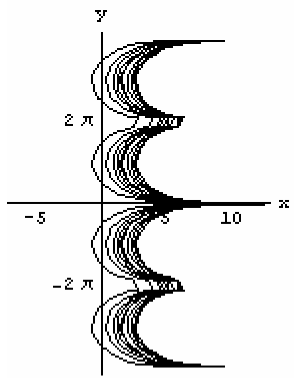




Finally the study was complemented with corresponding level curves:

ContourPlot[Exp[x]*(1-Cos[y]),{x,-7,11},{y,-11,11}];

where the information about minima was still lacking in the picture.

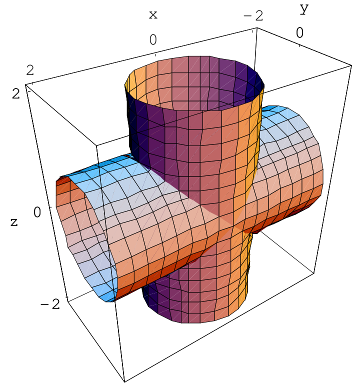


It was rewarding seeing these students finding all the information (knowledge previously acquired) they could think of, to push the new technologies into offering them the picture they required from having dealt with the analytical approach to the problem.

4. Integrals in 3D

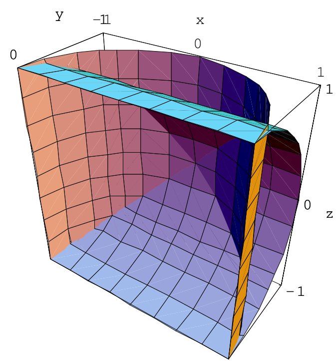
In order to help the students to calculate the common volume of two cylinders ($x^2+y^2=1$ and $y^2+z^2=1$) we gave them a picture which did not offer a clear view of the type of "container" (the intersection of the cylinders) they were supposed to be dealing with.

```
<<Graphics‘ContourPlot3D‘

p1=ContourPlot3D[x^2+y^2,{x,-2,2},{y,-
1,1},{z,-2,2},Contours->{1.}];

p2=ContourPlot3D[y^2+z^2,{x,-2,2},{y,-
1,1},{z,-2,2},Contours->{1.}];

Show[{p1,p2}];
```



Therefore a vertical cut was required, and the picture came out as

```
<<Graphics‘ContourPlot3D‘

p1=ContourPlot3D[x^2+y^2,{x,-1,1},{y,-
1,0},{z,-1,1},Contours->{1.}];

p2=ContourPlot3D[y^2+z^2,{x,-1,1},{y,-
1,0},{z,-1,1},Contours->{1.}];

Show[{p1,p2}];
```



The truth is that the students could not trust such pictorial information (now they had doubts about the image that was offered by the computer); they questioned the appearance of these plane sections because they did not believe that such curved surfaces could, by intersection, offer rectangular sections (squares).

The discussion, once again led the students into proving by means of analytical/algebraic manipulation what the computer was showing (and this time the visual information was correct, even if it was not predictable).

Conclusions

The main purpose of the experiment was definitely achieved: the students were much more involved within the semester work on the course and the final evaluation marks were, as never before, a success. The statistics are as follows*:

Academic Year	Registered Students	Evaluated Students	Passed/ Registered	Passed/ Evaluated
1999/2000	393	147	22.1%	59.2%
2000/2001	421	137	21.4%	65.7%
2001/2002	**438**	**166**	**29%**	**76.5%**
2002/2003	430	178	17%	42%

During the academic year (2001/2002) during which the experiment took place, the lecturers heard students who had no chance to follow the course that time wondering if from then on we would always be in charge of the course and would run it similarly; they had heard their colleagues commenting in favour of the course, which also showed a change in the typical students' attitude towards Calculus courses. Unfortunately, for reasons related to departmental policies, the lecturers had to run different courses the year after the experiment took place.

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* The headings in the table refer to terminology that is familiar in the Portuguese University system. They stand for:

i) **Registered Students:** Number of students (1st, 2nd, 3rd, 4th, and 5th years) who were officially registered for the compulsory course on *Multivariable Calculus*.

ii) **Evaluated Students:** Number of students who have, in fact, attended the course and undertook the assessment.

iii) **Passed/Registered:** Percentage of students who got a pass, out of the registered students (whether attending the course or not).

iv) **Passed/Evaluated:** Percentage of students who got a pass, out of the students who attended the course and undertook the assessment.

In summary, we believe that, using *Mathematica*® with these students and within the context of the course on Multivariable Calculus:

- was, as a tool for discussing ideas or as a resource for counter-examples, essential in developing logical/analytical reasoning as well as for implementing habits for justifying the results, which are fundamental to learning Infinitesimal Calculus;
- engaged the students in mathematical reasoning and motivated them to the necessity for algebraic/analytical approaches; this could be complemented with the symbolic manipulation offered by *Mathematica*®;
- improved substantially the pass rate in the course, without having altered the parameters or the level of demand of the examination.

As a result of this empirical study it is clear that there is scope for much more detailed research on the relationship between the use and outputs of the technology and the students' conceptual learning and mathematical behaviour in the area of Multivariate Calculus.

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