# Topic Subgroup 12

# Research and Development in the Teaching and Learning of Calculus

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In preparing for this working group, the organisers made a call for papers across a wide range of possibilities, including, but are not confined to: the role and use of technology in the teaching and learning of calculus; the role and use of history in the teaching and learning of calculus; research about cognitive process in the learning of calculus; pre-calculus; from calculus to mathematical analysis (or) transition between secondary school and university; contextual approaches; graphic approaches; didactical engineering; misconceptions and epistemological obstacles in the learning of calculus.

The papers received covered a wide range of topics from which seventeen were selected for presentation on the ICME website with eight having short presentations in the topic group, preceded by two introductory lectures and a final session ending with a one hour round table summarizing session.

The papers as a whole covered a range of different approaches to calculus for different target groups. Several papers focused on the nature of the limit concept, which in turn focused on the distinction between calculus and mathematical analysis. Most papers made reference to the use of new technologies, either through their increasing use in teaching or through research on their effects in learning. Several papers built didactical theories which were used to formulate the curriculum and as a basis for analysing the practices of learning and teaching.

To frame our discussions the organisers therefore focused on three sets of questions to be kept in mind during the lectures and to form the basis of the final round-table debate. These questions were as follows:

1) *How can we differentiate the teaching of calculus or mathematical analysis according to the target public?* What means and criteria are important to realize a particular approach adapted for a specific public? How do we address the difficulty of the initial limit concept? Is it helpful to see the first approach to calculus through a blend of embodied meaning and symbolism and to postpone formalism to a study of mathematical analysis. How do we deal with the problematic transition between calculus and mathematical analysis?
2) *What is the role of technology?* How can we characterize and categorize more deeply the ways of using new technologies to teach calculus and mathematical analysis? How do we take into account its use as a mathematical tool to solve problems, as a means of delivering the curriculum, and as a cognitive environment for learning?
3) *What do the various didactical theories bring to structure the preceding questions?* How do we evaluate the uses of new technologies in the learning of the students? What about the teacher's practices? How can we evaluate effects of these practices?

**Summary of presentations**

In the opening lecture, David Tall presented a framework for mathematical thinking that distinguishes three different modes of operation: the embodied world, “based on our sensory experiences and characterized by thought experiments,” the symbolic world, “based on our use of symbolism to carry out calculations and manipulations,” and the formal world that “relates to the building of formal theories based on definitions and proofs.” This framework was used to formulate the growth of ideas in the calculus, including two significant discontinuities: the shift from finite processes in arithmetic and algebra to the potentially infinite limit concept and the shift from embodied thought experiments and symbolic calculations to quantified definitions and proofs. He suggested that the new technologies benefit the symbolic world by "performing calculations and symbol manipulations at a level of accuracy that would be difficult or impossible by hand." They also benefit the embodied world "in a more subtle way by providing an enactive interface (...) that allows the user to control and experiment with visual representations.. On the other hand, the formal world has the least benefit because of the gap between "the finite machine and the actual infinity of the theoretical limit process."

Isabelle Bloch and Maggy Schneider presented a Francophone view using the theory of the didactical situations of Guy Brousseau to analyse questions about learning and teaching calculus and analysis. They revealed a progression from situations to help students to contrast a view of mathematics about instantaneous velocities and curvilinear areas to manipulation of quantifiers in the production of graphs of functions that satisfy given conditions. Through the use of three connected dialectics—action, formulation and validation—these situations formulate an analysis that has links with two or more of the three worlds theorized by David Tall. One important criterion which emerges to analyse new trends of teaching is the learning by adaptation created by an appropriate milieu (a term for the context of learning introduced by Brousseau): the students have to construct concepts mobilized in questions beyond a usual didactical technique showing the objects of the knowledge.

Deploring the frequent lack of a proper but difficult approach to real numbers in high school or in university level calculus and the implicated misconceptions of the students, Talma Leviathan from Israel proposed "a project of a transitional course centred around number systems, where the real numbers wouldn't be introduced through an abstract set-theoretic definition but rather through the concrete notion of decimal expansion". This project articulates a geometric approach by "filling the gaps" in the "rational numbers ruler" and a more abstract approach where a real number is defined as a decimal expansion which one associates its infinite sequence of truncations. So the classical properties of the real numbers can be rigorously proved.

Laure Barthel presents a computerized interactive module that familiarizes the students with decimal expansions of real numbers. With the help of this module, students are encouraged to investigate the properties of decimal expansions (e.g., periodic ones) to have a concrete introduction to several issues concerning the notion of limit.

Michel Helfgott, who teaches calculus at college level in the USA, presented and illustrated five guidelines which appear to him important in the teaching of first-year calculus:

- Try to strike a balance with regard to what to prove and, what to accept without proof. For example, we can rely on geometric intuition in the case of the mean value theorem and prove an unexpected result such as the derivative of the product of two functions.
- Convey the idea that sometimes there is more than one way to solve a problem; for instance, to calculate some integrals by substitution or integration by parts.
- Discuss significant applications in the classroom, not relegating them to the end as optional materials; for example, problems about kinematics, optimization, work and so on.
- Place the subject in a historical perspective whenever possible, like summations linked to problems of integration across history.
- Use technology to supplement mathematical learning, not to supplant it; for example, having students learn how to build programs related to Newton's and Euler's method of approximation.

Victor Giraldo and Luiz Mariano Carvalho from Brazil described a qualitative study of the effects on learning caused by an approach to differentiation based on the embodied idea that a differentiable function, when magnified locally, 'looks straight'. Their approach uses computer technology including software called *Best Line* and the symbolic manipulation software *Maple* to study both simple cases in which the graphic picture is a good representation of the numeric processes and also cases where finite computer arithmetic is compromised to produce pictures that fail to fit the expected theory of limits. The conflict between the finite world of the computer and the perfection of human thought experiment is used to enrich the formal meanings with a suitable pedagogical approach.

Erhan Bingolbali from Turkey showed the influence of departmental affiliation of students (first-year undergraduate mechanical engineering and mathematics) on their developing conceptions of the derivative, based on several kinds of data: quantitative (pre-, post- and delayed post-tests), qualitative (questionnaires and interviews) and ethnographic (observations of calculus courses and `coffee-house´ talk). The findings reveal that mechanical engineering students develop a tendency to focus on rate of change while mathematics students develop a tendency for tangent-oriented aspects. He argued that this difference cannot be solely attributed to the practice of the courses that the students had followed. He further suggested that department affiliation appears to have influence on cognition and plays a crucial role in the emergence of different tendencies between the two groups.

Yury Shestopalov and Igor Gachkov from Sweden described their method for using mathematical software in courses based on the real-time-mode teaching for students at university. Demonstrations and computations are performed directly in the classes using TI calculators, PC, desktops, workstations. The authors focused on an example concerning interpolations with natural splines. The main argument which supports this approach is as follows: the use of the CAS programs is often reduced to pure illustrations of computing process and is unsuitable for non-conventional mathematical courses which gain increasing popularity and which become unmissable for the modern engineer, such as coding theory, discrete mathematics and scientific computing.

Isabelle Bloch and Imene Ghedamsi bring to the fore factors of rupture between the secondary mathematical organisation in teaching (pre-)calculus and the university approach to the case of the concept of limit. Their theoretical references are various: tools introduced by Aline Robert to distinguish different functionalities of the limit and several sorts of knowledge; the anthropological theory of Yves Chevallard and modelled in terms of tasks, techniques, technologies, theories; the distinction made by Anna Sfard between procedural and structural approaches; and the notion of semiotic representational settings of Raymond Duval. These theoretical frames allowed the authors to identify main didactical variables which measure the very important rupture between the two levels: the degree of formalisation, the setting of validation, the degree of generalisation, the number of new notions introduced in the limit environment, the type of tasks, the choice of techniques, the degree of autonomy of the students, the process or object status of the concepts and the nature of the transitions between the semiotic representation settings.

Elfrida Ralha, Keith Hirst and Olga Vaz present a form of a cooperative learning using *Mathematica* in the teaching of multivariable calculus for first-year Informatics' students at university in Portugal. The size of the traditional classes (200 in lectures and 65 in tutorials) was not changed, but a new methodology was introduced in tutorial classes, partitioning the allocated time into "students working on their own", "students sharing solutions" and "the lecturer summarizing the fundamental ideas for the session". Some of the treated examples offered the students the opportunity to realize misconceptions that can arise from graphics and to express their conceptual doubts. The aim is to engage them in mathematical/(productive) reasoning and to motivate them to the necessity for algebraic/analytical justification. A qualitative study and statistics show among other things a change in the typical students' attitude towards calculus and an improvement in the pass rate for the course.

Salahattin Arslan and Hamid Chaachoua talked about the dominance of an algebraic approach to teaching differential equations in the upper secondary school in France and the lack of numeric and qualitative study. They formulated the hypothesis that the limitation to only the algebraic frame for the treatment of differential equations can be the origin of difficulties and habits which students faced with qualitative interpretation tasks, for example:

- difficulty to recognize the slope of the tangent can be calculated from the differential equation;
- difficulty to recognize a differential equation, in particular non-linear ones.

The authors described a module on differential equations set up within a framework of training teacher-trainees to test this hypothesis, among some things. They explained reasons for which one can hope to gain insight from software, in particular a dynamical one such as *Cabri*, because it can provide students with opportunities to discover new resolution tools.

In the final discussion session, Michael Thomas, David Smith and De Ting Wu led with their responses to the presentations framed within the initial three questions: relating to the variations in approach that may be appropriate for different target groups, the role of technology and the role of theoretical frameworks.

Michael Thomas focused on the role of technology and the various theoretical frameworks, asking whether technology should be used to teach familiar ideas better or to teach new ideas appropriate to the new situations made possible by technology. He suggested that research and development to date has been either driven by practice or by theory but that there are signs that the two approaches are converging.

David Smith focused initially on the question of different approaches for different target groups and suggested that in his own university the decision was taken that there were only three major concepts to be studied: rate of change, accumulation, and the relationship between the two, so that all students should attend the same course, with the possible exception of mathematics major who may require a more comprehensive understanding of analysis. He suggested a new theoretical framework by Kolb that emphasized learning activities that start with Concrete Experience, then proceed through stages of Reflective Observation, Abstract Conceptualization, and Active Experimentation. He related this to a neurophysiological theory of Zull that linked the way in which the Kolb cycle of learning relates to the use of different parts of the brain. He concluded by speaking about his own ways of using computer technology to teach the calculus.

De Ting Wu presented his professional opinion based on teaching the best of the old college calculus and using new technology where this was appropriate. He acknowledged that technology is powerful tool and a helpful aide in teaching and learning but affirmed that it would not replace the study of mathematics or the value of good teaching.

In the closing discussion, Bronislaw Czarnocha questioned the major emphasis in the discussion on the cognitive difficulties in understanding calculus and emphasized the power and beauty of the subject. Maggy Schneider argued that it was necessary to subordinate the analysis of teaching using new technologies to more general theoretical frameworks about learning and teaching relating to the nature of the school institutions and the teachers' practices.

The subgroup provided a fruitful sharing of many ideas, with Anglophone and Francophone theories meeting in a constructive manner and research from many parts of the globe showing that the study of calculus remains vigorous and creative in many different ways.