Research situations for teaching: a modelling proposal and example

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Scientific knowledge is built in the context of research, especially through solving open questions. This observation led us to study how didactic processes could exist and work around research situations. The training involved is concerned primarily with "transverse" knowledge, i.e. those which play a role in many scientific fields, such as experimentation, conjectural statements, argumentation, modeling, definitions, proofs, implications, structuring, decomposition-recomposition, induction. However, we observed that, one the one hand, learning this transverse knowledge is a constant objective which has been declared "essential" after several program reforms in secondary education in France, and that, on the other hand, there is an intrinsic difficulty to carry out these objectives in class.

The types of situations which we analyze here have been worked out for a long time in various workshops, at all school levels, and have been studied from a theoretical point of view in the last three years by the SIRC group, composed of researchers from various departments and teachers from secondary education.

I. Research situations for teaching : a modelling

I.1. Hypotheses and research questions

In order to make progress in a research situation, a researcher can, and must often, select by himself a suitable framework of resolution, must modify the rules or allow himself to redefine objects or questions. This is precisely this type of practices which we wish to get pupils involved with, because they are the foundation of mathematical activity. However it seems that this type of practices is not usual in class, and even that it is practically forbidden in many circumstances.

This raises the question of finding which conditions are needed in didactic institutions to create a mathematical activity which pertains to a "research situation", and is likely to allow the learning of what we have called "transverse" knowledge.

I.2. A model of "Research Situation for the Class" (RSC)

For us, an RSC must fulfill the following criteria (These criteria will be developed in the session).

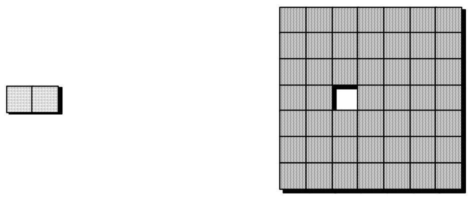
1. *A RSC is akin to a professional research strategy*. It must be related in some way to unsolved questions. Because, there is a strong argument that a close contact to unsolved questions, not only for the pupils, but also for teachers and researchers, will be decisive for establishing the pupils' positioning with respect to the situation.
2. *The initial question should be of an easy access*. In particular, the question can be easily understood by pupils only when the problem does not require heavily formalized mathematics.
3. *Possible initial strategies are in view*, without requiring specific prerequisites.
4. *Several research strategies and several developments are possible*, from the point of view of mathematical activity (construction, proof, calculation) as well as from the point of view of mathematical concepts involved.
5. *A solved question can possibly lead to other new questions* .

I.3. Illustration of this model on a particular situation

This situation (Grenier & Payan (1998)) has been tested for several years, from the CM1 level (9 year old pupils) to University (DEA, i.e. first year of Post Graduate studies) and is now integrated regularly in various teaching curricula.

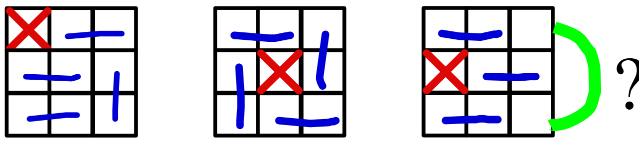
The proposed problem is a paving problem, namely paving a certain domain of boxes by pieces, without overlaps or overflows beyond the limits. More precisely, the question consists of knowing whether a given "polymino" can be paved by copies of identical smaller polyminos. In this generality, it is an open question which stands no chance of being solved. Researchers are currently interested in rather particular cases, for instance, whether it is possible to pave subsets of a square grid by smaller polyminos.This question is of an easy access, even for very young pupils.

Pupils and teachers first agree on a starting point of research, for example the following question : can one pave with dominos, a square grid from which a single cell has been removed (this cell being arbitrary). Here is an example as it could be shown to pupils.

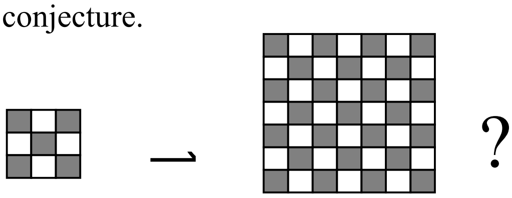


Initial strategies exist. It appears sufficient to have an intuitive feeling of space allowing to identify a set of cells and to understand what is a paving; this knowledge is already available at the nursery school level. The concept of parity is of course involved, but it is not essential, in fact the situation is a tool for investigating and understanding it in more depth.

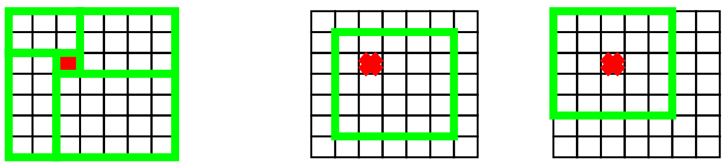
Devolution of the problem is immediate: some testing of pavements around the initial example allows to get a grasp on the question, without solving it completely. When the point is to advance further in the problem, it becomes necessary to change the stated question: one will for instance work with smaller squares (3x3 or 5x5). When pupils do not manage to pave, they can notice that in order to cover certain cells some choices are forced; one therefore obtains a proof of impossibility through "forced choices".



There are various strategies for progressing in the research. The location, on a 3x3 grid (and also on a 5x5 grid), of positions of cells which are to be removed to allow paving (represented in gray on the figure), induces a more general conjecture.



Evidence on the possibility of paving is given by various ways of partitioning, among which inductive procedures play a role ("inductive partitioning"). Examples :



Proofs of impossibility are obtained by structuring the object, while exploiting the form colored by the "right" boxes: since a domino covers a "black" cell and a "white" cell, a pavable polymino must necessarily be "balanced" (i.e. it must contain as many black cells as white boxes in a "check board" coloring).

The situation is not a "dead end": in fact, the phase of resolution of the initial problem (paving of a square grid with one cell removed) has raised in a natural way further questions.

I.4. Knowledge and environment concerned by a RSC

Our experimental research confirms that there is actual training involved, and that these are constitutive of any mathematical research activity: arguing, conjecturing, structuring objects, proving, modeling - all these items are more or less involved according to the selected RSC.This is what gives an institutional legitimacy to these situations.

The elements of the triple (question, conjecture, proof) are the "invariants" of the RSC. The associated "didactic variables" are "variables of research", in the sense that they determine the understanding and the interest of the question, its suitability for opening new questions, broadening research strategies, transforming the problem (modeling).

The criterion of success for pupils is not only, as in usual exercises, to solve the question (whether the solution is right or false). A "partial" criterion of success can be that pupils have raised a strong conjecture, or simply solved a particular case. The criterion of success for the teacher is the recognition of progress in the area of procedural knowledge (question, conjecture, proof).

I.5. Position of the actors in the didactic situation comprising a RSC

In a RSC, the actors (pupils and teachers) are in positions which differ from the usual ones of traditional didactic situation.

• Pupils are in a researcher position because they are assigned the task of producing something "new" which is not only new for them. Our experimental data show that, for pupils, the fact of knowing that they are trying to solve an unsolved, or only partially solved, problem, modifies their approach to the activity.

• Teachers are in the combined position of researchers and managers of the situation. As researchers, their position is closer to the pupils than in a traditional situation. But they are (supposedly) possessing the required transverse knowledge and the evaluation criteria for their validity. The "institutional relationship" between pupils and teachers is indeed concerned with this transverse knowledge. The corresponding basic rules are the usual ones occurring in "scientific debate" (Legrand, 1993).

II. An experimental situation

Part of our research is focused on studying research situations presented in the form of games[1], and introduced using suitable material support. We make the assumption that such a presentation can be a help with the devolution of the problem, already at the primary school level. Thus, through experiments which we carry out, we try to answer the following questions:

What is the role of the material gaming medium in the devolution of the research situations ?
What can be the influence of this medium on the research strategies ?
How can one manage a RSC presented in the form of a material game ?

II.1. An example: the wheel with colored pawns

Formulation of the task.

A fair organiser proposes a game made up of two discs of different sizes, laid out in a concentric way. On the largest disc, he displays a certain number of pawns, all of different colors.

Rules*:* The player must place on the small disc the same number of pawns as on the large disc. These pawns can use one, two, three, four or more colors, selected among the colors laid out on the large disc by the organiser. The small disc turns, notch by notch. The player wins if, in each position of the small disc, one and only one of its pawns is of the same color as that which corresponds to it on the large disc. How can the player choose and lay out his pawns to win? [2]

Elements of resolution

We will not attach a complete resolution of this problem, because the reader will be interested in it

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[1]in the sense that one, two or several « players » can play together, that possible actions are organized under rules (precise instructions), and that games are based on the use of some sort support, whether it is a material support, some data-processing or paper-pencil work. The « gaming environment » makes it possible to orient certain or all aspects of research situations in the direction where they can present problems in particular cases (under some choice of values of variables).

[2] G.Polya and Mr. Gardner in particular have studied this problem.

in order to get an idea of what it means to investigate such a problem. The pair (n, k) constitutes a variable of research. According to its value, the *progress* that can be made to solve the problem will be different. Indeed, the values of this variable can be classified in two categories, which correspond to different formulation and validation phases:

- case where there are several solutions: in this case, the formulation and the validation will consist in producing particular solutions, possibly supplemented by general methods of construction.

- case where there is no solution: solving the problem will consist in formulating the conjecture "there is no solution", and in validating this conjecture by means of mathematical arguments or by means of an exhaustive search of cases.

In addition, in order to progress in this problem, it is necessary to detach oneself from the actual colors and to consider the relative position of the sectors (or representing pawns), when compared to the others. That allows us to reach general methods of construction. One can introduce an additional variable, namely a shift, which can be defined differently according to the values of the pair (n, k). For example, in the case of the pair (n, n), the shift measures the position on the external disc with respect to the interior disc. One can thus obtain arguments of proof in the case where no solution exists.

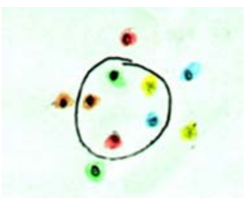
II.2. Conditions of experimentation

At the time we set up our experimentation, we made several choices. First, we chose to ask pupils to work *by groups of 3 or 4*. In addition, we provided the necessary *material equipment* to each group, in the form of two metal discs and magnets of different colors in sufficient number.

As the activity was planned in several sessions, we also gave to each group a *sheet of assessment* on which the pupils could record at any time the results of their research when they thought they were important, so that they could later rely on their written notes. Moreover, after several research sessions running under this setup, we organized a *joint session* with all groups, so that they could communicate and pool their results, their methods, their conjectures, and possibly hold further discussions. During this final meeting, the groups did not have the material equipment at disposal. There was only one available for the whole the class on which the colors were not chosen by the pupils, but which they could use to illustrate their matter.

We posed the same problem to students of 1st year of university, 11/12 and 9/10 years old pupils. The mathematical situation was managed by a researcher, while the teacher ensured the social management of the class.

II.3. Analysis of pupils' productions



Taking our observations into account, we consider that, except for two groups of 9/10 year pupils and one group of 11/12 year pupils which remained at the very first stage of the game, all the others entered a mathematically oriented, more or less elaborate, step of research.

At all levels, several values of the pair (n,k) were studied. Moreover, no outstanding differences appeared in the research dynamics of the different groups, although the school levels were different, except perhaps for an « inductive » method of research[3] which only appeared at the primary level. Several other methods of constructions of solutions have been proposed, whatever the level and studied cases were. They appear to be inspired by the research strategies. In the three secondary classes, the shifts were introduced by several groups. Even if they did not have organized steps of research, all the groups raised some conjectures. Some proceeded by grouping conjectures associated with particular cases ; others tended to raise more general conjectures. Among them, all managed to state the conjecture that there did not exist a solution whenever that was the case. However this appeared more easily at the university level. We make the assumption that this difference is possibly due to a more prevalent

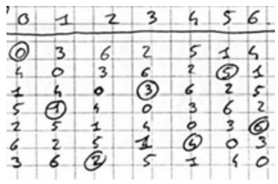
[3] It consists of keeping what works fixed and modifying the rest.

idea among pupils of secondary levels that a problem always has a solution, considering that this is true for the majority of the exercises which are proposed to them. At university, students can be confronted with exercises without solutions, e.g. in the case of the resolution of equations, and our students had in any case be faced to such issues through research situations they had studied beforehand. Finally, except for three university groups, all other groups relied on an exhaustivity approach (which was not a priori taken for garanted) in order to prove the absence of solution, and used very little proof arguments in those cases.

This situation thus worked quite well as a RSC within the directions that we defined. The training concerned corresponds to those situations which we previously evoked, such as to consider that a problem of mathematics does not inevitably have a unique solution, in fact it can have several ones or none at all. Other perspectives considered were to "decontextualize" the situation, working out methods of construction, seeking to generalize, but also stating conjectures, invalidating them by counterexamples, asking the question "why ?".

II.4. About material support and installation conditions

It appears that the existence of a material support facilitates the problem devolution. It also enabled the pupils to provide counterexamples whenever necessary, in order to give a basis to their conjectures. We thus make the hypothesis that the material support is a help with research because it permits a more direct handling.



However, even if the construction of methods of resolution that were discovered are similar, their formulation seems to have been influenced by the use of the material support. Among the university groups, those who used it at the beginning of their research quickly proceeded to paper-pencil work, introduced a numerical coding of the colors and a representation of the problem with a table. They gave methods of construction which tended to be detached from the concrete description, by using a mathematical vocabulary and by trying to generalize.

As we had supposed, the fact of working in group allowed students to debate, argue and avoid discouragement. Moreover, the game approach seems to enable pupils in difficulty to develop their argumentation, as they were led to discuss with other pupils which usually had better success in mathematics. They seems possible, ultimately, because this type of problems puts pupils at an "equal level of knowledge". The assessment sheets seem to help the pupils to judge what is important or not, and avoids them getting lost from one week to the other. Moreover, they show the importance of the clearly setting down what is going on, for the sake of re-using later the data. The joint session allowed the different groups to orally formulate their construction methods, which they had not necessarily succeeded to do in their written work. The fact that the choice of colors is not the pupils responsibility made it possible to invalidate research methods by gropings, to the profit of more organized research methods which were developed because they are more effective.

The presentation of one RSC using a suitable material support reinforces its accessibility at all levels, by facilitating the problem devolution. However, when the aim is to lead pupils to decontextualize and to generalize, it appears necessary to propose at least a session without any material support provided, or, at the very least, during a work sesssion for which it is not of any substantial help with research. We currently study how this "withdrawal" must be negotiated and what consequences can be attained from such attempts. In addition, we try to take into account the time variable and to establish under which conditions the repeated practice of RSC as a shared activity allows pupils to learn the various components of research activity in mathematics we underlined, and can influence their personal viewpoint with respect to mathematics.

References

Arsac G. (1990) Les recherches actuelles sur l'apprentissage de la démonstration et les phénomènes de validation en

France, Recherches en didactique des mathématiques, vol. 9/3 pp.247-280, ed La Pensée Sauvage, Grenoble.

Audin P., Duchet P. (1992) La recherche à l'école : Math.en.Jeans, Séminaire de Didactique des Mathématiques et de l'Informatique. n°121, pp. 107-131, Grenoble 1.

Duchet P. (1990) "Maths en Jeans", International Conference on Combinatorics, Marseille-Luminy.

Godot K. (2003), Situations recherche et jeux mathématiques: premières analyses, actes de la 12 ème école d'été de didactique, Corps, 2003.

Grenier D. et Payan, C. (2003) Situation de recherche en classe : essai de caractérisation et proposition de modélisation, cahiers du séminaire national de recherche en didactique des mathématiques, Paris, 19 Octobre 2002.

Grenier D., (2002), Different aspects of the concept of induction in school mathematics and discrete mathematics, European Research in Mathematics Education, Klagenfurt, august, 23-27.

Grenier D. (2001), Learning proof and modeling. Inventory of fixtures and new problems. Actes du 9th International Congress for Mathematics Education, ICME 9, Tokyo, Août 2000.

Lakatos I. (1976). Preuves et réfutations. Paris : Hermann Ed., 1985.

Legrand M. (1993) Débat scientifique en cours de mathématiques et spécificité de l'analyse, Repères IREM n°10, pp.123-159. Topics edition.