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Children's Visual Imagery: Aspects of Cognitive Representation in Solving Problems with Fractions [1]

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ABSTRACT

We outline and discuss briefly some aspects of internal, cognitive representational systems in the characterization and analysis of mathematical problem solving. We then focus on visual imagistic representations inferred from a fine-grained qualitative analysis of two elementary-school children solving problems involving fractions. The problems, designed to require conceptual interpretation in relation to rational numbers, were presented during two scripted, videotaped task-based interviews administered 16 months apart. We discuss the children's concrete/pictorial imagery, pattern imagery, and memory images of formal notation, in the context of representations based on set models (discrete quantity) and spatial extent models (continuous quantity). We further discuss internal operations on and transformations of internal imagistic configurations.

Keywords: Visual Imagery, Cognitive Representations, Mathematical Problem Solving, Qualitative Analysis, Fractions.

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[1] Parts of this report draw on the doctoral dissertation research of Adrian DeWindt-King (2002), performed under the supervision of Gerald Goldin at Rutgers University. We extend here an earlier presentation by the authors at the 25th Annual Conference of the International Group for the Psychology of Mathematics Education (DeWindt-King and Goldin, 2001).

SYSTEMS OF REPRESENTATION AND MATHEMATICAL PROBLEM SOLVING

Mathematics is often defined as the science of pattern. Patterns occur in nature, of course, but they occur also in the representational systems that we human beings have devised to describe our experiences of nature. For example the development of the system of "natural numbers", together with a means of numeration for representing them, permits us to encode and record the results of counting processes applied to real-life objects. But once the system of numbers is established, the study of patterns in that system becomes itself a subject of mathematical inquiry. Likewise the development of the "rational numbers", together with a notational system for representing them ("fractions"), allows us to encode and record quantitatively part-whole relationships, linear and area measurements, comparisons of quantities, and so forth. But the study of the properties of the rational numbers, as a system in its own right, then also becomes part of mathematics.

A pattern, whether in the natural world or in a mathematical system, can be regarded from one point of view as *inherent.* That is, the pattern is present—it is "there to be discovered"—whether or not any particular person at any particular time has identified it, described it, or internalized it. We often take this view when we teach mathematics: we have some patterns in mind that we hope to lead our students to discover and understand through appropriate activities and discussion. From another point of view, a pattern can be regarded as something that is *constructed from, brought to or imposed on* the natural world or a mathematical system by a cognizing, interpreting individual or social group. We often take this view when we focus on describing students' conceptions, especially when these may be nonstandard or partially developed. Frequently it is important to consider the *interaction* between the internal cognitions of the individual, and the external patterns or structures in the educational or task environment.

These considerations apply to the discussion of *representational systems* in mathematical learning and problem solving. A *representation* is any configuration (of characters, images, concrete objects, etc.) that can denote, symbolize, or otherwise "represent" something else (Palmer, 1978, Kaput, 1985, Goldin, 1987, 1998). Such representing relationships are often two-way, so that the depiction or symbolization can be interpreted in either direction (Goldin & Kaput, 1996; Vergnaud, 1998). Thus the fractional notation "3/4" can refer to a diagram in which a circle is partitioned into four parts, three of which are shaded. Alternatively, depending on the context, the diagram can be considered to represent "3/4". As the examples of natural numbers and rational numbers suggest, individual representations do not generally stand in isolation. They belong to systems that are characterized by rules and procedures (possibly with ambiguities) for forming representational configurations, and by higher structures that relate the configurations within a system. Thus we can properly consider base-ten Hindu-Arabic numeration, fractional notation, Cartesian graphs, or standard algebraic notation, as among the conventionally-established representational systems of mathematics. Where we say one representational system leaves off and another begins is to a large degree a matter of convenience and convention.

We emphasize the important distinction between systems of representation that are *external* to the individual (including the above-mentioned conventional systems of mathematics), and the *internal* psychological representations of the person. This distinction is needed to discuss patterns inhering in the task environments of individuals (external), patterns constructed in or brought to mathematical situations by individuals (internal), and the interactions between the external and the internal that occur during mathematical learning and problem solving. One cannot, of course, *observe* the internal representations of others. In teaching mathematics (and in daily life), we must make *inferences* about other peoples' internal representations, based in part on their interactions with or productions of external representations. The present article is about such inferences.

Earlier efforts by one of us to develop a model for mathematical problem-solving competency structures and their development led to a description based on five types of systems of internal representation (Goldin, 1987, 1992, 1998, 2000a, Goldin & Kaput, 1996). These are: (1) *verbal/syntactic systems* describing competencies related to natural language; (2) *imagistic systems*, including visual-spatial, auditory-rhythmic, and tactile-kinesthetic encoding; (3) *formal notational systems* of mathematics, as internalized by the individual; (4) a *system of planning, monitoring, and executive control* that guides strategy use and executive decision-making during problem solving; and (5) an *affective system*, including not only general, "global" affect, but also the changing states of feeling (emotion) that can occur (and encode important information) during problem solving. These system types are not isolated but mutually interacting, so that internal representations of one type represent and evoke representations of other types while problem solving is taking place. In this model, it is clear that solving problems in mathematics involves far more than writing down and manipulating standard symbolic notations.

The model proposes to describe the development of representational systems over time through three major stages: (a) An *inventive/semiotic* stage. Here *meaning is first assigned* to internal configurations, with reference to previously established representations (Piaget, 1969). (b) A period of *structural development.* In this stage the construction of relationships within the new system is driven by or structured with reference to the initial assigned meanings. (c) An *autonomous* stage. The developing representational system is no longer attached through its initial meaning to prior representations, but functions flexibly with new meanings and in new contexts.

Psychologically, learners often think of the initial meaning as the "real" meaning of the new configuration—for example, when the fractional notation "3/4" is first introduced to mean "three out of four equal parts of a whole object," this may become what a fraction "really is" to the child. Many properties can be developed based solely in this interpretation, including addition of fractions with like denominator (where the sum remains less than 1), multiplication of fractions less than 1, and the equivalence of fractions where one denominator divides the other, such as "2/3" and "4/6".

As the structural development continues, the initial meaning is strengthened. But other, important mathematical aspects of the rational number system are not supported very well by this interpretation—for example, rational numbers greater than 1. Such aspects may consequently pose conceptual barriers for learners. Detailed consideration of the early stages of representational development helps us understand such potential sources of students' cognitive obstacles (see also Goldin & Herscovics, 1991; and Goldin & Shteingold, 2001).

The new representational system eventually acquires greater power as the learner builds other semiotic connections. Thus rational numbers eventually come to "mean" quite flexibly ratios, operators, points on a number line, multiplicative inverses, and so forth.

Comparable stages seem also to describe the historical development of mathematical ideas, and lie behind the characterization of such obstacles as "epistemological."

SOME ASPECTS OF IMAGISTIC REPRESENTATION

Various researchers have considered imagistic representation generally, and visual imagery in particular, as a fundamental cognitive system of representation for mathematical learning and problem solving (Bishop, 1989; English, 1997; Goldin, 1982, 1987, 1998; Goldin & Kaput, 1996; Owens, 1993; Presmeg, 1985, 1986, 1998; Thomas & Mulligan, 1995; Thomas, Mulligan, & Goldin, 2002). Here we focus on concrete/pictorial imagery, pattern imagery, memory images of symbolic notation, dynamic imagery, and mental operations on or transformations of images. These characteristics are not, of course, mutually exclusive—for example, pattern imagery may be dynamic, and pattern images may be transformed *via* mental operations.

We use the following definitions, which we consider to be consistent with the usage of Owens and of Presmeg.

Concrete/pictorial imagery refers to internal imagery spoken of, gestured at, or represented externally as if it were a physical object or a picture. The image may be given a name referencing what it resembles in real life. Presmeg (1985) found pictorial imagery to be the predominant form during problem solving. Owens also found it to occur frequently, especially during the orientation phase of problem solving, and again as a kind of retrospective, global check.

In *pattern imagery* the image abstracts and/or generalizes mathematical relationships. Presmeg (1985, p.175) notes, "such imagery may be vague or vivid but its essential feature is that it is pattern-like and stripped of concrete details." Krutetskii (1976) calls pattern imagery a graphic scheme where the problem's terms and relations are represented in a visually schematic way.

Memory images of symbolic notation refer to the visualization of formal mathematical expressions. Presmeg, for example, reports students who describe "seeing" a mental picture of a formula, and "reading" the image.

In *dynamic imagery* (as contrasted with static imagery) there is some indication or report of active movement or change in all or part of the image (Thomas & Mulligan, 1995).

Mental operations or transformations refer here to visualized purposeful acts by the imager that modify or transform the image. This suggests, but not require, dynamic imagery, as transformations can also be visualized as a succession of static images. Presmeg also uses the term “kinaesthetic imagery” here, in connection with associated physical actions. Owens’ term is “action imagery.”

As our study of imagery is centered in the domain of rational numbers (Behr, Lesh, Post, & Silver, 1983; Behr, Wachsmuth, & Post, 1988), we consider the children’s imagery in relation to various possible conceptual models of fractions. These include discrete (set) models as distinct from continuous quantity or spatial extent models (area or volume, sometimes termed “region models”); and part/whole models as distinct from comparison or ratio relationships. We consider mental operations such as partitioning. We also focus on children’s understandings of fractional symbolic notation through visual imagery.

QUESTIONS MOTIVATING THE STUDY, AND DESIGN OF THE INVESTIGATION

Our study is exploratory. The questions that motivated it are the following.

(1) *Inferring internal imagery.* Recognizing that internal imagistic representations are constructs to be inferred from observable (external) behavior, can we create explicit criteria for inferring the five characteristics of visual imagery mentioned above from hand gestures, body movements, paper and pen or marker diagrams, pictures, and charts, and physical manipulations of concrete materials? Do we then find consistency among these in the individual child?

(2) *Relation between internal and external imagistic representation.* What relationship can be discerned between the child’s internal visual representational capability and his or her facility in constructing external imagistic representations? How does the use of paper and pencil or marker to draw pictures, diagrams, and/or create charts, or the use of concrete manipulative materials, assist in or influence the visual imagery used in the problem solving?

(3) *Relation between internal visual imagery and strategy.* Heuristic strategies for problem solving that are of interest to us include drawing diagrams, working backward, solving simpler or similar problems, guess and check, subproblem decomposition, and so forth. How does the child’s internal imagery interact with or influence strategy use?

(4) *Relation between internal imagery and inference of mathematical patterns.* How is the child's facility in constructing or recognizing a mathematical pattern influenced by his or her imagery? Are particular characteristics of imagery important here, and if so which ones?

(5) *Internal imagistic representation and mathematical conceptual development.* Is there a relation between the child's internal imagery and the development of his or her conception of a fraction over time?

We make use of task-based interview methodology. This involves videotaping and observing individual children interacting with or producing external representations as they solve problems posed by the clinician. From our observations of their behavior, we draw some inferences about their internal cognitive representation. But such research has major limitations. The construct of internal representation entails ambiguity, while the process of inferring internal representation requires interpretations that are context-dependent. As we are reporting partial results from an exploratory and descriptive "status study," we do not commit ourselves to creating a definitive coding scheme whose reliability might be tested. But for the research to be eventually generalizable, we do set the goal of making our observations, and our criteria for making inferences from them, as explicit as possible. Thus we aim to improve the task-based interview methodology as we explore individual children's imagery.

As part of a longitudinal study of individual children's mathematical development, five structured task-based interview scripts were developed. Some general scientific principles for designing planned, structured task-based interviews in mathematics education research motivated the design. The formulation of principles was also influenced by the experience gained in the study.

Ten such principles have been discussed in elsewhere in detail (Goldin, 2000b). Briefly, they are as follows: (1) Design the interviews to address research questions formulated in advance, not just to "see what occurs." (2) Choose tasks that the children can represent meaningfully, but which pose challenges or difficulties. (3) Let the tasks embody rich and varied representational structures, with mathematics that can be characterized through meaningful semantic relationships and through visual imagery, as well as abstractly. (4) Provide in as much detail as possible a description of the interview protocol, with scripts that contain explicit criteria for the major expected contingencies, to facilitate the comparison and possible replication of findings. (5) Encourage free problem solving to the maximum extent possible. (6) Maximize the opportunities for interaction with the external learning environment, providing a variety of external representational possibilities. (7) Decide in advance what will be recorded, and record as much as possible. (8) Train the clinicians. and pilot-test each interview outside the main study to identify unexpected contingencies. (9) Build alertness to new or unforeseen possibilities explicitly into the design. (10) Make conscious, appropriate compromises when conflicts occur among scientific principles.

Additional detail about the design of the five interview scripts has been presented elsewhere (Goldin, DeBellis, DeWindt-King, Passantino, & Zang, 1993; Goldin, 1997). They were developed at Rutgers by a team of experienced teachers that included the clinical interviewers in the study, and revised after rehearsal and pilot testing to incorporate many different contingencies. A total of 22 children, ages 8 to 10 years at the outset of the study, were selected to participated in five task-based interviews over three school years.

Interviews #2 and #5 involved non-routine problems about and with fractions (see Carpenter, Fennema, & Romberg, 1993, and Litwiller & Bright, 2002, for various perspectives). A broad, cognitive analysis has been completed of all 20 children who participated in both of these interviews, with respect to the fraction representations used by the children and with respect to the children's strategies (Passantino, 1997; see also Goldin & Passantino, 1996). The present article reports on a portion of a study focusing in greater depth on the visual imagery of a cross-sectional subset of four children (DeWindt-King, 2002). We shall discuss excerpts from interviews with two of them, "Londa" and "Marcia," inferring some contrasting characteristics of their visual imagistic representations.

In accordance with the above task-based interview principles, free problem solving is encouraged throughout each interview. After posing a question, the clinician allows for *spontaneous* responses before asking questions to explore the child's answer, request the child to explain using external representational materials. However if the child does not seem engaged in the task, or appears at an impasse, pre-planned suggestions or hints are offered. The clinicians do not impose their methods or correct or confirm the child's solution, as it is *not* the goal here to teach rational number concepts. At certain points, decided in advance and described in the interview script, the children are guided toward particular understandings that are essential for subsequent questions to be meaningful to them.

Two video-cameras operated simultaneously during each interview. One is focused on the child's work, while the other shows the interaction of the clinician with the child. Outlined next are elements of the two interview scripts needed for our discussion here. The complete scripts may be found in Passantino (1997), or are available from the authors.

In task-based interview #2, materials placed before the child include a pad, pencil, markers, and red and black chips (checkers). Some preliminary, non-mathematical questions precede a sequence of mathematical questions posed by the clinician. For *each* of these mathematical questions, the follow-up includes, as appropriate, the queries, "Can you help me understand that better?" "Why?" and/or "Are there any other ways to take one half (one third)?"

The first such questions in interview #2 are:

- "When you think of one half, what comes to mind?"
- "When you think of one third, what comes to mind?"
- "Suppose you had twelve apples. How would you take one half?"
- " ... How would you take one third?"

Cutouts are then presented, in succession: a square, a circle, and a 6-petal flower. For *each* the child is asked:

- "Here is a shape. How would you take one half?"
- "... How would you take one third?"

Part way through the interview, some retrospective questions are posed, with follow-up:

- "What was on your mind when you were answering the questions up to this point?"
- "Did you have a picture in your mind while you were answering any of the questions?"

The interview continues:

- "Can you write the fraction one half?"
- "What does this fraction mean to you?"
- "Can you write the fraction one third?"
- "What does this fraction mean to you?"

The clinician then goes on with more complicated exploratory activities, that involve taking one half and one third of an array, and visualizing the cutting of a cube into fractional parts. At the end of interview #2 retrospective questions again address the child's visual imagery.

Task-based interview #5 is designed to explore again specific topics from interview #2, and extends to further rational number problems. Materials placed before the child at the outset include red and white chips, paper circles, squares, and triangles, markers, pencil and paper, a calculator, a ruler, ribbon, scissors, and other items specific to later questions. The follow-up for each main question includes, as appropriate, the following queries: "Why?" "Why not?" "Can you show me what you mean?" "Can you show me [using] the materials?"

The main questions begin as follows:

- "When you think of a fraction, what comes to mind?"

Bold-face printed expressions in vertical format for the five fractions, 1/2, 1/3, 2/3, 3/4, and 4/6, are presented on a sheet of paper:

- "What fractions do you see here?"
- "Can you explain ... what one of these fractions means?"
- "Why is it written this way?"

Several main questions later, the clinician asks:

- "Imagine a big birthday cake shaped like a rectangle. Can you imagine what it looks like?"
- "Describe what it looks like."
- "Now imagine that there are 12 people coming to the birthday party and they each want a piece of cake. Your job is to cut the cake so that each person gets the same size piece. How will you cut the cake?"

...

- "Are there any other ways to cut it?"
- "Now think about the icing. Suppose the cake has icing on the top and side."

...

At the end of the interview, further retrospective questions are asked that address the child's recollections of visual imagery.

OBSERVATIONS AND INFERENCES: LONDA

For each of the two children discussed here we shall provide a brief overview discussion, followed by more detailed excerpts from the two interviews.

At the time of interview #2 Londa was 9 yrs. 8 mos. old, and attending the 4th grade of a school in a low-income, urban district.

Her spontaneous representations of "one half" and "one third" are verbal: *"a cookie, split in half" "... and then when you split it in half, it's two ... Two smaller pieces,"* and *"Half of our classroom, kids in our classroom," "... we have 19 kids in our classroom so half of that would be 9 ... no, we have 18, half of that would be 9,"* followed by *"box cut in threes ... because I think a box would be easy to picture," "one-third of an orange,"* and *"one-third of an apple. Really one-third of anything."*

These verbal phrases are all suggestive of internal concrete/pictorial 3-dimensional imagistic representational configurations, drawn from familiar real-life contexts. Interpretation of the fraction seems based on a mental operation of partition of a whole into equal parts. All but one image provide a "spatial extent" or continuous quantity model; the "kids in our classroom" can be characterized as a "set" or discrete quantity model.

Asked to take one-half of 12 apples, Londa replies, *"Split all of the apples in half."* Showing with checkers, *"Well, take 12 of these"* [takes 6 red, 6 black] *"I would split them all in half. So then there'd be 24"* [she counts the 12 checkers, assigning a value of two to each] *"'Cause you cut this in half this would be two ... four, six, eight"* [she continues to 24]. When asked to take one-third, Londa suggests, *"Cut them all into threes"* [she counts by threes with the 12 checkers, to 36], but she never explicitly states what would be one-third.

Londa's external representations to this point suggest mental operations visualized through a succession of static, 3-dimensional concrete/pictorial image-configurations, using almost exclusively a continuous quantity model for "fraction." She does not come back to a "set" model. Having acted to transform her image from 12 apples into 24 pieces, she does not seem to treat the latter as equivalent elements. Rather she retains the pairing derived from the apple at which each pair originated.

Asked to write the fractions one-half and one-third, Londa does so correctly in a vertical format, symbolic notation. Interpreting what she wrote, she explains, *"This* [the 2] *means that you have two parts, and this* [the 1] *being one whole. One whole and two parts."* She explains one-third as *"one whole, three parts."* [Clinician:] "Is there any other way you can think of what those fractions mean?" [Londa:] *"Either that or three wholes and one part."* She explains, *"if I had one cookie, two cookies, three cookies"* [draws three circles], *"and I take one cut, part out of each of them"* [makes a small wedge at the top of each circle].

Londa translates consistently, but with some instability, among verbal expressions, pictorial images with mental operations, and memory images of symbolic notation. Her interpretation of the numerator as representing the whole and the denominator as the number of parts is consistent with her first answers, and with her method of taking one-half and one-third of twelve apples. She does not show evidence of a ratio or comparison model for fractions.

At the time of interview #5 Londa was 11 yrs. 0 mos. old, attending the 5th grade of the same school.

In this interview, her initial statement of what comes to mind for a fraction is: *"I think about how the denominator means that it's the whole and the numerator means how many parts are out of it." ... "Like two thirds, the three is the denominator, that means the whole thing; and two means how many, um, two pieces out of the three whole."* [Clinician:] "Could you show me using some of these materials perhaps?" [Londa:] *"Okay, this is two,* [picks up two yellow cutout circles] *this is the two-thirds, and then you have three on the bottom, you have three all together* [puts a third circle with the others] *and then you have this is two* [indicates the two circles] *and then you have one left over* [indicates the third circle]. *So, well, like, if you say two-thirds minus, um, one-third you have two-third, no, two-thirds minus one-third you have one third and that's it."* [without further action with the circles]

Londa's first words suggest a memory image of symbolic notation that is related directly to part-whole imagery. Her use of the circles makes no further reference to "pieces of the whole," but suggests a possible relation to a set model, even the genesis of a ratio model, but with additive imagery. She also evidences a memory image of an algorithmic procedure for subtracting fractions.

Later in the interview, asked to visualize the rectangular birthday cake, Londa describes it, and explains: *"Since it's a rectangle you can divide it into equal parts."* She shows its height with the ruler perpendicular to the table, *"It's about two inches thick,"* and draws a rectangle with two rows and six columns to show exactly how to cut it in 12 equal pieces. Asked if there are other ways, she replies, *"I don't think so because that's the way we cut my birthday cake. I,* [shakes her head] *I don't think so."* Asked how many, and then what fraction, of her pieces would have icing on exactly two sides and the top, she (correctly) explains *"Just the four, because … you can't put icing on the inside of a cake …"* and *"one-third, well, four-twelfths you can reduce it, four-twelfths can be reduced by four and it'd be one-third."*

Londa's verbal and external pictorial representations of how she cut the cake, and her way of determining the number of pieces with icing on exactly two sides and the top, suggest static, internal 3-dimensional, real-life pictorial imagery, with operations of partition and counting closely embedded in the context. Londa uses her imagistic part-whole model and an internal algorithm to obtain the desired fraction of the pieces.

We find in these two interviews evidence of the possible nature of the development over time of Londa's fraction concept, and changes in her use of visual imagery. The progression seems to be from: (a) predominant use of pictorial imagery and mental operations on the pictorial image, and (b) an initial, strong commitment to a part-whole, region model for fractions in real-life contexts, where the "whole" is the individual object; to: (a) pattern images as well as pictorial images, together with memory images of symbolic notation, with (b) some conceptual understanding of both region and set models for fractions, and extensive use of the region model to compare sizes of fractions, as well as algorithmic procedures for manipulating fractions formally.

We close this section with three comparisons of excerpted interview segments, which provide additional detail. The diagrams are schematic depictions of Londa's drawings.

Comparison 1: In interview #2, all but one of Londa's spontaneous representations suggest an internal pictorial image of a region model for fractions drawn from a familiar real-life context. The exception is her representation of half of the children in her classroom, which is still pictorial in a real-life context, but can be characterized as a set model for the fraction as an operator, with an additive structure. In contrast, in interview #5 Londa evidences an internal memory image of symbolic fractional notation and a memory image of an algorithmic procedure for subtracting fractions, meaningfully associated with the pattern imagery of a set model for fractions.

Interview #2

Clinician: When you think of one-half, what comes to mind?

Londa: Um, a cookie, split in half.

Clinician: Okay. Um, can you tell me more about that?

Londa: Uh. When I think of one-half I think of a cookie because that's the first thing that comes to mind, and I don't know. I think it's just because a cookie and then when you split it in half, it's two.

Clinician: Mmm-hmm [nods].

Londa: Two smaller pieces.

Clinician: Okay. Um, is there any other way you can think of one-half?

Londa: Half of our classroom, kids in our classroom.

Clinician: Mmm-hmm [nods]. Ah, can you tell me more about that?

Londa: Um, we have 19 kids in our classroom so half of that would be nine.

Clinician: Ah.

Londa: No, we have 18, half of that would be nine.

Clinician: Uh-huh. Uh-huh, and, um, can you explain that a little more to me? Why would it be the half at nine?

Londa: Because nine plus nine equals eighteen.

Clinician: Okay. When you think of one-third, what comes to mind?

Londa: [Pause.] Uh, box cut in threes.

Clinician: Okay, and why do you think of that?

Londa: Because I think a box would be easier to picture...

Clinician: Mmm-hmm [nods].

Londa: ... in threes.

Clinician: What kind of a box?

Londa: Uh, just a regular box. Like something that you would put things in to ship.

Clinician: Mmm-hmm [nods].

Londa: Just something like that.

Clinician: Okay. Is there another way you could think of one-third?

Londa: Mmm [shakes head] ... uh ... that would be easier?

Clinician: Just any other way.

Londa: One-third of an orange.

Clinician: Okay [nods].

Londa: One-third of an apple.

Clinician: Okay [nods].

Londa: Really one-third of anything.

Interview #5

Clinician: Good. Tell me, what do you think of when, um, well when you think of a fraction, what comes to mind.

Londa: Um, well, I think about how the denominator means that it's the whole and the numerator means how many parts are out of it.

Clinician: Can you show me what you mean?

Londa: Umm ...

Clinician: You can use any of these materials.

Londa: Like two-thirds, the three is the denominator, that means the whole thing; and two means how many, um, two pieces out of the three whole.

Clinician: Okay. Could you show me using some of these materials perhaps?

Londa: Umm. Okay, this is two [picks up two yellow cutout circles], this is the two-thirds, and then you have three on the bottom, you have three all together [takes a third yellow cutout circle and places it with the other two], and then you have this is two [pointing to two yellow cutout circles] and then you have one left over [pointing to the other, third, yellow cutout circle]. So, well, like, if you say two-thirds minus, um, one-third you have two-third, no, two-thirds minus one-third you have one third and that's it.

Comparison 2: In interview #2, Londa uses symbolic notation to write the fractions one-half and one-third, and utilizes part-whole region and set models to explain the meaning of the fractions. She draws three circles, *each* with a wedge, to explain the meaning of the fraction one-third. She is able to translate from verbal to symbolic notation, but is unsure of the interpretation of the symbolic notation. In contrast, in interview #5 Londa uses spoken symbolic notation to identify a series of proper and improper fractions, and utilizes a set model or a pictorial region model to correctly explain the meaning of the fractions, and to compare them (not always correctly). She describes the set model and draws, then shades in, circles and rectangles to explain the meanings of the fractions.

Interview #2

Clinician: Okay. Can you write the fraction one-half? [The clinician hands Londa a card.]

Londa: One-half. [Picks up paper and writes:]

$$\frac{1}{2}$$

Clinician: Yeah. What does this fraction mean to you?

Londa: This [points to the two] means that you have two parts and this [points to the one] being one whole. One whole and two parts.

Clinician: Mmm-hmm. One whole and two parts. Mmm-hmm. Can you write the fraction one-third? [The clinician hands Londa another card.]

Londa: [Picks up paper and writes:]

$$\frac{1}{3}$$

Clinician: And what does that mean to you?

Londa: One whole, three parts.

Clinician: Mmm-hmm. Mmm-hmm. Is there any other way you can think of what those fractions mean?

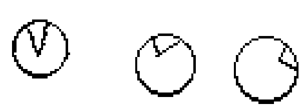
Londa: Either that or three wholes and one part.

Clinician: Mmm-hmm. Can you explain what that means?

Londa: Mmm. That you have three whole pieces and you're cutting out one part.

Clinician: Can you show me what you mean?

Londa: Uh, say if I had one, two cookies, three cookies. [Londa draws:]



Clinician: Yeah.

Londa: And I take one cut, part out of each of them.

Interview #5

[Londa is presented with a pink sheet of paper with the following fractions written: 1/2, 1/3, 2/3, 3/4, 4/6.]

Clinicians: Londa, what fractions do you see there?

Londa: One-half, one-third, two-thirds, three-fourths, and four-sixths.

Clinicians: Can you explain to me what one of those fractions means?

Londa: First of all four-sixths is an improper fraction.

Clinicians: Why is that?

Londa: Because.... Oh, no, I'm sorry, I was thinking of six-fourths. Um, you can make them smaller. Like four-sixths, you can divide, um, two into four and that would be two and two into the six and that would be three and you have two-thirds. Um....

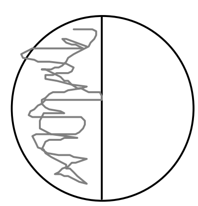
Clinicians: Can you explain to me what one of those fractions means?

Londa: One-half means you have two pieces, two, um, two, two of the whole and one means that you take one, that you have one, like, you have one-half you have one-third and you divide it into two equal parts and you shade in one and that's one-half.

Clinicians: Okay. Why, why is it written this way?

Londa: So that you know what's the divider and what's the numerator? The denominator is the number underneath the line and the numerator is the number above the line. [coughs]

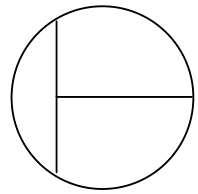
Clinicians: What you have explained to me about what one-half means, could you show me, using some of these materials?



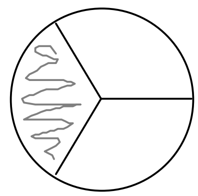
Londa: Mmm-hmm. Take this, and you can have one whole thing and you divide it into two so you have two parts and you shade in one and that's one-half. [Londa draws a circle with a vertical line down the middle and shades the left half.]

Clinician: Okay. Which fraction is the smallest in, in the group?

Londa: Um, in order to do that I would do this, I didn't have, you would draw it into circles, and you just divide it into three. It's not gonna be even, I can't draw equal. [Londa draws a circle with a vertical line from 11:00 to 7:00 and a horizontal line beginning at the center of the vertical line and extended to 3:00.]

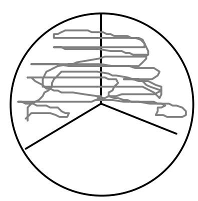


And it's gonna be something like this, [redraws the circle with a sideways Y (3:00, 7:00, 11:00)] no that's four, no, one, two, three, yeah, [counting the sections in the circle with sideways Y] and you shade in one [shades one third from 7:00 to 11:00].



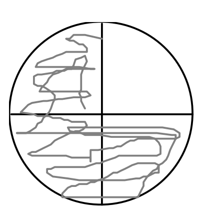
And this piece [referring to the drawing representing one-half] is bigger than that [referring to the drawing representing one-third] so you know that so far this [one-third] is the smaller piece.

And then you draw two-thirds. You would have the same thing and you divide it, oh [she mistakenly draws the circle with the vertical line from 7:00 to 11:00, so



she redraws the circle, this time with a upside down Y], you and divide it into three parts and you shade in two [shades the upper two thirds].
And this piece [the two-thirds] is bigger than that piece [the one-half] so, so far this [the one-third] is still the small piece [coughs].

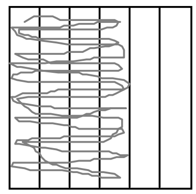
And then you draw a circle and divide it into four [draws a circle with a horizontal



and vertical line], and you have three [shades 3:00 to 12:00],

So … so far that's the bigger piece now [the three-fourths], so that's still the smaller piece [the one-third].

So now you take this [draws a circle] and you do it into, and I'll do a square this time [draws a square], and you do it this way, one, two, three, four, five [counts as she draws vertical lines] and you shade in four one, two, three, four [shades four of the six sections]; and you have two left over. So the one-third is the smallest.



Clinician: Which fraction is the largest in the group?

Londa: Four-sixths.

Clinician: Why?

Londa: Because this area [pointing to the inside of the square representing four-sixths] is bigger than this area [the circle representing three-fourths], this [the four-sixths] is the bigger area of these two [pointing to the circles representing four-sixths and three-fourths] and that area [pointing to the square representing four-sixths] is bigger than this area right here [pointing to the circle representing three-fourths].

Clinician: And why?

Londa: [Coughs.] Um, because you have more pieces, more, they're divided into more groups.

[Londa is presented with a blue sheet of paper with the following fractions written on it: 5/5, 3/1, 5/4, 11/8, 10/8.]

Clinician: Okay. Londa, what fractions do you see here?

Londa: Five-fifths, three-ones, five-fourths, eleven-eighths, and ten-eighths, and they're all improper fractions.

Clinician: Can you explain to me what this [points to the fraction 5/5] fraction means?

Londa: It means that, um, out of the five wholes we have five and you can reduce that to one, because that'd be, the whole thing would be shaded in, so that's one whole thing.

Clinician: Okay.

Londa: [Coughs.] Um, for these [refers to the other fractions on the blue paper] in order to make them a mixed fraction you divide the denominator into the numerator and whatever you have left over that becomes your numerator you keep this in the denominator. Like this [3/1]would become one goes into three two times so it would be, and you'd have one left over so it'd be two and one one and you could do it again and it'd be three because one and one equal one and it'd be three. This one [points to 5/4], four goes into five one time and you have one left over so it'd be one and one-fourth.

Clinician: Could you show me what you mean for that one?

Londa: Um, okay. This you'll have five-fourths [writes “5/4=”], and it equals, because this is an improper fraction. You know it's an improper fraction 'cause the numerator is higher than the denominator so you divide the denominator into the numerator and it equals one [writes “1"], four minus, um, five equals one so you have one left over and that becomes your numerator [writes 1 of 1/4] and you keep the same denominator so it becomes one and one-fourth [writes the /4]. [The final equation looks as follows:]

$$\frac{5}{4} = 1\frac{1}{4}$$

Clinician: Okay.

Londa: And you do the same thing to the rest of them.

Clinician: Okay, could you do those last two for me?

Londa: [Coughs.] Sure. Um, eleven-eighths equals, eight divided into eleven would be one and you have three left over so it would be one and three-eighths and you can't reduce it. [Londa writes as she talks.]

$$\frac{11}{8} = 1\frac{3}{8}$$

And um the last one would be that, and that eight divided into ten is one and you'll have two left over so it'd be two-eighths and you can reduce that to one, and you divide two into each of them so two divided into two is one and two divided into eight is four. [Londa writes as she is talking.]

$$\frac{10}{8} = 1\frac{2}{8} = 1\frac{1}{4}$$

Clinician: Looks kinda' like a nine doesn't it [pointing to the 4 in 1 1/4]? [Londa fixes the number to look more like a four.] Londa, which fraction is the smallest fraction in the group?

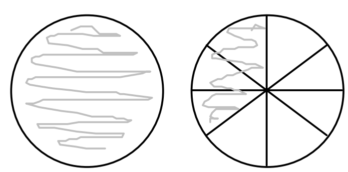
Londa: [Coughs.] I'd say five-fifths because that equals one whole and the rest of them they have a whole number and a fraction in front of it, so that's....

Clinician: Okay, which fraction is the largest fraction in the group?

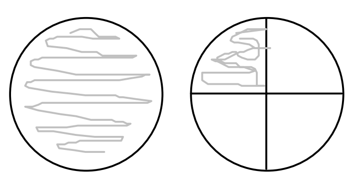
Londa: Um, I say three over one because it equals three wholes and the rest of them all the whole numbers are only one and the whole number for that one is three.

Clinician: Are there any fractions in the group that are the same size?

Londa: I think these two might be [points to 11/8 and 10/8 on the paper she just wrote on]. If you have, okay, you have one whole and you shade the whole thing in [draws a circle and shades the whole circle] and you have another one [draws another circle], [coughs] you have this would be four [divides it into four sections] and then you do it again [bisects each section to make eighths] and it'd be eight one, two, three, four, five, six, seven, eight, and you shade in three of them [shades from 7:30 to 12:00] and it'd be like this.



And if you have one and one fourth you shade in the whole thing [draws a circle and shades the whole circle] and you divide it into four and you shade in one [draws another circle with a horizontal and vertical lines, shades one section from 9:00 to 12:00]. No [shakes head] that's not it [Londa sees that her drawings are not equivalent]. Uh ...



Yes, five fourths [on the blue paper] and ten over eight [on her own paper which shows that 10/8 = 1 2/8 = 1 1/4] because you reduced that to one and one-fourth and this comes out to one and one-fourth, too [pointing back and forth]. That's it.

Clinician: There aren't any others?

Londa: [Coughs and shakes head.] No.

Comparison 3: In interview #2, when Londa is asked to take one-half and one-third of twelve apples she suggests cutting each apple into two or three pieces, respectively. She explains that if you cut each apple in half there would be a total of twenty-four pieces and if you cut each apple into three there would be a total of thirty-six pieces. Londa attempts to use the checkers to demonstrate her point. She counts out twelve checkers, then counts by twos, then by threes. Her external representations suggest mental operations visualized through a succession of static, 3-dimensional concrete/pictorial image-configurations, using exclusively a continuous quantity model for "fraction." She does not come back to a "set" model; having acted to transform her image from 12 apples into 24 pieces, she does not seem to treat the latter as equivalent elements but retains the pairing derived from the apple at which they originated. In contrast, in interview #5 when Londa is asked to cut a cake into twelve equal pieces, and then to determine the number of pieces with icing on exactly two sides and the top, her verbal and external pictorial representations of how she cut the cake, and her way of determining the number of pieces with icing on exactly two sides and the top, suggest static, internal 3-dimensional, real-life pictorial imagery, with operations of partition and counting closely embedded in the context. Londa uses an imagistic part-whole model, and an internal algorithm, to obtain the desired fraction of the pieces.

Interview #2

Clinician: Uh-huh. Hmm. Suppose you had twelve apples.

Londa: Mmm-hmm [nods].

Clinician: How would you take one-half?

Londa: Split all of the apples in half.

Clinician: Ah-ha. Um, can you show me what you mean?

Londa: Mmm. With these [points to checkers]?

Clinician: If you'd like to.

Londa: Well, take twelve of these. [Counts twelve checkers and separates them from the group.]

Clinician: Mmm-hmm.

Londa: And I would split them all in half. So then there'd be twenty-four.

Clinician: Okay. Um, could you explain to me why there would be twenty-four?

Londa: 'Cause you cut this [touches one checker] in half this would be two...

Clinician: Mmm-hmm.

Londa: ...this would be four, six, eight, ten, twelve, fourteen, sixteen, eighteen, twenty, twenty-two, twenty-four. [Londa touches the checkers as she counts.]

Clinician: Mmm-hmm. Okay. Suppose you had twelve apples again. How would you take one- third?

Londa: Cut them all into threes.

Clinician: Yeah.

Londa: Then there would be; three, six, nine, twelve, fifteen, eighteen, twenty-one, twenty-four, twenty-seven, thirty...thirty-six. [Londa touches the checkers as she counts.]

Interview #5

Clinician: Okay. Londa, imagine a big birthday cake, shaped like a rectangle.

Londa: [Coughs.]

Clinician: Can you imagine what it looks like?

Londa: Mmm-hmm. [Nods.]

Clinician: Can you describe that to me?

Londa: It has four sides and all the sides are equal. Um, the two lengths are equal and the two widths are equal.

Clinician: Can you tell me a little bit more about this picture?

Londa: Umm.

Clinician: A bit.

Londa: Since it's a rectangle you can divide it into equal parts. [Coughs.]

Clinician: About how, how tall is the cake?

Londa: Oh! How tall is it? It's about, let's see, take the ruler [holds the ruler vertically, with the end touching the table], let me see. It's about two inches thick. Two inches thick.

Clinician: Okay. Now imagine that there are twelve people coming to the birthday party and they each want a piece of cake.

Londa: Mmm-hmm.

Clinician: You're job is to cut the cake so that each person gets the same size piece.

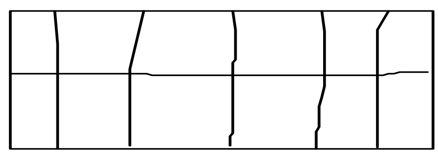
Londa: Okay.

Clinician: How will you cut the cake?

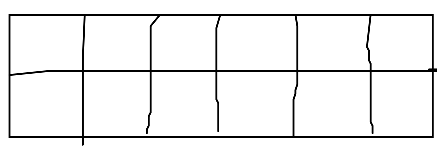
Londa: You divide it in the middle, you make a, um, horizontal line, [coughs] and then you slice it vertical until you get, um, twelve parts. You have six on each row.

Clinician: Can you show me what you mean?

Londa: Um, okay, you gotta go like this, like that, like that and like that, and you slice it like this; and you go one, that's two, four, six, eight, ten...not equal. [While Londa talks, she draws a rectangle, with a horizontal line and five vertical lines].



Oh. [Londa talks as she redraws the cake, to make each piece equal]. First you have to slice it like this, then you do it like this, like this, like this, like this; and



then you have one, two, three, four; you have eight [two rows of four], ten and twelve [draws the last vertical line to make the last four sections].

Clinician: Okay. Are there any other ways to cut it?

Londa: [Coughs.] Um, [shakes head] I don't think so, because that's the way we cut my birthday cake. I, [shakes head] I don't think so.

Clinician: Did you have twelve people?

Londa: Mmm-hmm. [Nods.]

Clinician: Yeah? Now, think about icing. Suppose the cake has icing on the top...

Londa: [Coughs.]

Clinician: ...and on the sides.

Londa: Mmm-hmm.

Clinician: If you cut the cake the way you said to cut it, how many pieces of the cake would have icing on the top and exactly two sides?

Londa: One, two, three, four [indicating the corner pieces]; because this side would have icing, and this side would have icing, this side have icing, this side would have icing, this and that, that, that, and plus on the top, and these slices won't have any. These will too because they're on the side, 'cause we icing the cake ...

Clinician: Which ones will too, did you say?

Londa: All these on the outside. These.

Clinician: Uh-huh.

Londa: All of them. On two sides?

Clinician: Yes.

Londa: Oh, no. [shakes head] Okay. I made a mistake. Just the four, because we iced it on the outside and this side and that's two sides of the cake and plus on the top. You've the same thing with this, the same thing with that, the same thing with that; these will have it, because this side of the cake, or this side with that side, that's the inside of the cake, so you can't put icing on the inside of a cake unless you have the filling as icing.

Clinician: Okay. What fraction of the pieces have icing on the top and two sides?

Londa: [Coughs.] One-third, well, four-twelfths you can reduce it, four-twelfths can be reduced by four and it'd be one-third.

Clinician: And why do you say that?

Londa: Because these are the only ones that have icing and that's four...

Clinician: Mmm-hmm.

Londa: ... and out of the whole that's twelve, so you have four-twelfths, and you reduce it, 'cause you can, it's not the lowest term.

OBSERVATIONS AND INFERENCES: MARCIA

At the time of interview #2 Marcia was 10 yrs. 4 mos. old, in the 5th grade in a small, lower middle income school district.

Her spontaneous, verbal representations of "one half" and "one third" are *"a half of a circle," "a half of a triangle,"* and *"a third of a circle."* These suggest possible internal pattern or pictorial imagery with a region model for the fraction. Asked about the 12 apples, Marcia replies, *"Count 'em. Well if you know there's 12, and 6 ... and since 6 and 6 is 12, take 6 of them if you want half."* [arranging 12 red checkers in two rows of 6, and counting them] *"... you know that 6 and 6 is 12 you could just take half of them away and you would have half."* [She pushes the 6 checkers from the top row into a casual irregular group. Clinician:] "Why is it one half?" [Marcia:] *"Because, well, if you place it like this and you add the 12 to your 6 here"* [she arranges the two rows of 6 again, indicating that taking the bottom row of checkers is equivalent to taking the top row, and counts] *"... it's the same amount the other way."* For one-third of twelve apples Marcia replies, *"if you knew 4 times 3 was 12, you could take 4 away and you would know it was a third ... because 4 and 4 is 8 and then another 4 is 12."* [pushes 4 checkers to the right, then separates the remaining 8 into two groups of 4, maintaining the row and column structure]

Marcia's external representations suggest memory images of notation, and additive and multiplicative structures for part-whole relation in a "set" model. She focuses on the numbers of apples, doing some mental computation. Importantly, her physical manipulation of the checkers suggests an internal pattern image that includes operational one-to-one correspondence between items in parallel rows.

Later Marcia correctly writes the fractions one-half and one-third in vertical-format symbolic notation, explaining, *"well, I can tell that it's one-half because like this is a two* [points to denominator] *and it's a one* [points to numerator] *and if you add one and one it would equal two,"* and *"if you could times one times three or one and one and one is three so it would be one-third."*

Here Marcia's repeated addition of the numerator to reach the denominator value is suggestive of her earlier pattern imagery.

At the time of interview #5 Marcia was 11 yrs. 8 mos. old, in the 6th grade in the same community. She now describes a fraction in quite general terms, *"like a part of a whole of something"* [writes the fraction three-fifths] and explains, *"like three-fifths it's like the five-fifths is the whole, and it's like that's part of the whole of whatever the thing is."*

To us this suggests not just internal memory images of notation, but also a level of abstraction that includes a pattern image of a fraction as an operator.

Marcia initially responds to the birthday cake question verbally, kinesthetically, and with a drawing: *"Well, um, it's like this* [tries to indicate with her hands how the cake would look, then draws a triangle] *like, a, uh, oh a rectangle"* [draws a long, thin vertical rectangle]. After some dialogue she measures and redraws a horizontal rectangle 4 inches long. *"... and like you would divide the four into half ... which is at two* [draws a vertical line at 2 inches, dividing the rectangle in half] *and then you would divide each of these into halves* [draws vertical lines dividing each section in half] *... so you'd have four pieces ... and then you divide these, which is at the half an inch, 'cause the other one was at the inch to divide ... and then you have to divide these which is at a fourth of an inch"* [counts 16 pieces]. ... *"But there's more than twelve so ..."*

Prompted to redraw freehand, Marcia suggests, *"it would depend on like how big the cake is 'cause like if the cake is, like, um, 24 inches long then you would have to divide them each into two inches" ... "So, it depends how long the cake is."* Further encouraged to redraw freehand, Marcia again considers halving, *"... there would be 16 ... since there's 8 now ... so it would double it so like you ..."* [Clinician:] "Yeah, could you do it?" [Marcia:] *"... would have to divide the cake into thirds"* [draws a long horizontal rectangle, with vertical lines dividing it in thirds] "*... and then divide these so then there's 6* [draws vertical lines dividing each of the 3 sections in half] *and then you divide these into half* [draws additional vertical lines dividing each of the six sections in half] *which would make that 12 pieces."*

Marcia does not think of other ways to cut the birthday cake into 12 equal pieces. Asked how many, and then what fraction, of her pieces would have icing on exactly two sides and the top, she answers (correctly), *"there would be 10, 'cause there, it would be on the top here* [points to interior of the rectangle] *and then it would be on this side and this side* [points to the upper and lower horizontal edges] *but you can't count these sides* [points to the right and left vertical edges] *'cause there would be three since there's the edges."* She continues, *"10 out of 12, or five-sixths," explaining, "you divide the 10 and the 12 by two and you get five-sixths."*

From this and other evidence we infer internal visual and kinesthetic pattern imagery based on linear and region models, interacting with the "trial and evaluate" halving strategy. We think that Marcia's linear model stems from her measurement process.

From interview #2 to interview #5, we infer development of Marcia's pictorial imagery and pattern imagery, and a progression toward memory images of symbolic notation and conceptual understanding of a region model and of algorithmic procedures for fractions.

We close this section with three further comparisons of excerpted interview segments, which provide additional detail about Marcia. The diagrams are schematic depictions of Marcia's drawings and arrangements of manipulatives.

Comparison #1: Marcia's spontaneous representation for a fraction goes from being suggestive of an internal pattern or pictorial image of a region model in interview#2 to being suggestive of an internal memory image of symbolic notation in interview #5. In interview #2, her spontaneous representations for the fraction one-half were half of a circle and a half of a triangle. Her spontaneous representation for the fraction one-third was a third of a circle. In contrast, in interview #5 Marcia's spontaneous representation for a fraction was "a part of a whole of something." When prompted by the clinician she wrote the fraction three-fifths and explained that it represented part of a whole, five-fifths.

Interview #2

Clinician: When you think of one-half what comes to mind?

Marcia: A half of a circle.

Clinician: A circle. 'Kay. Um, is there anything else that you think of? Is there another way to think of one-half?

Marcia: A half of a triangle. [Laughs.]

Clinician: [Laughs.] All right. Ah, and what about one-third, when you think of one-third what comes to mind?

Marcia: A third of a circle.

Clinician: Third of a circle, and, ummm, is there anything else? Is there another way you can think of one-third?

Marcia: [Pauses then shakes her head no.]

Interview #5

Clinician: Okay. Good. All right, now when you think of a fraction what comes into your mind?

Marcia: Um, like a part of a whole of something.

Clinician: Mmm-hmm. Um, could you show me what you mean using something here?

Marcia: Um. Like, like a fraction, like three-fifths it's like the five-fifths is the whole, and it's like that's part of the whole of whatever the thing is. [Marcia writes the fraction 3/5 on the paper in vertical format with a horizontal fraction bar.]

$$\frac{3}{5}$$

Clinician: Mmm- Okay, and uh, have you studied fractions in school?

Marcia: Mmm-hmm. [Nods]

Clinician: And what have you studied about fractions?

Marcia: Well, we learned how to multiply them, and add them and subtract them, and stuff like that; and we learned how to use them.

Comparison #2: In interview #2, Marcia uses symbolic notation to write the fractions one-half and one-third, and used an additive method to show that repeatedly adding the numerator would result in the denominator. In contrast, in interview #5 Marcia uses spoken symbolic notation to identify a series of proper fractions, and explains how the fraction four-sixths can be reduced to two-thirds using a multiplicative algorithm. She also explains that the fraction four-sixths written in a vertical format means there are six parts in the whole, and four of these parts are either shaded or not shaded.

Interview #2

Clinician: Ummm. Card here ... And on this card could you write the fraction one-half?

Marcia: Writes on card:

$$\frac{1}{2}$$

Clinician: And when you write that like that, what does that mean to you? When you write one-half.

Marcia: One-half.

Clinician: Laughs. Okay. Ummm. Does the symbol mean anything to you at all?

Marcia: Well, I can tell that it's one-half because like this is a two and it's a one and if you add one and one it would equal two. [Points to the one and two with her pencil.]

Clinician: All right. Uh, and what about, uh ... could, on this card, could you write the fraction one-third?

Marcia: Writes on the card:

$\frac{1}{3}$

Clinician: And what does that fraction mean to you? When you write one-third. What does that mean?

Marcia: Shrugs her shoulders.

Clinician: Hmm. Like, you explained what the one half meant...

Marcia: Well, like ...Ummm. If, if you could times one times three or one and one and one is three so it would be one-third. [Marcia uses her finger to point to the one and three.]

Interview #5

[Marcia is presented with a pink sheet of paper with the following fractions written: 1/2, 1/3, 2/3, 3/4, 4/6.]

Clinician: Okay. I have a paper here. All right, um, what fractions do you see here?

Marcia: Um, a half, a third, two-thirds, three-fourths, and four-sixths.

Clinician: Mmm-hmm. All right, could you tell me about one of those fractions?

Marcia: Um.

Clinician: Explain, explain what it means.

Marcia: Well, like a half [points to the fraction 1/2] is a half of the, like it'd be a half of this paper; and like two-thirds and four-sixths are [points to the fractions 2/3 and 4/6], like, the same, this is just in lowest terms. [Marcia writes the fractions 2/3 and 4/6 on paper.]

$\frac{2}{3} \qquad \frac{4}{6}$

Clinician: Oh yeah, could you show me what you mean?

Marcia: Like, um, two-thirds and then four-sixths, like to find that out, like, um, two times two is four and three times two is six.

Clinician: Mmm-hmm.

Marcia: So, like, they, like, go into each other.

Clinician: Mmm-hmm, and you're saying they're the same?

Marcia: And this [points to 2/3] is just, like, the lowest terms of this [points to 4/6].

Clinician: Mmm-hmm. Mmm-hmm. And, uh, uh, why, why do they write the fraction like that? Like, uh, why do they write it two and a line and a three; and a four and a line and a six?

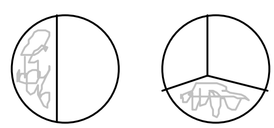
Marcia: Um, because, like, there's six parts in the whole and, like, four is like either how many are used or on a piece of paper, sometimes, how many are shaded in or how many aren't shaded in.

Clinician: Hmm. All right, uh, which fraction is the, um, is the smallest fraction in this group?

Marcia: [buzzer sounds in background] [Pause.] Um, one-third, I guess.

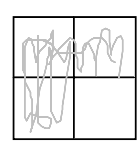
Clinician: Mmm-hmm, and could you tell me why that's the smallest?

Marcia: Um, well, because it's, two-thirds [points to 2/3] is more than one-third, because, like, two is more than one and a half, like, if you divided something in half [draws a circle with a vertical diameter] and then you divided something into thirds

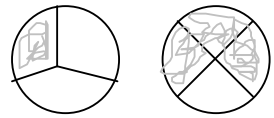


[draws a circle with an upside down Y] and um, like and you shaded the half in [shades left half], and shaded a third in [shades bottom third] um, more is shaded in when you shade in the half

And um, for three-fourths [draws a square divided into fourths with a horizontal and vertical line and shades all but the bottom right fourth], like, three-fourths would be shaded in and....



Um, wait. I guess I have to do it like this [draws a two circles one with an upside down Y and shades the left third, and the other with an X and shades all but the bottom one fourth].



Clinician: Why?

Marcia: 'Cause, um, it, 'cause it's hard to divide, um, this [points to the square] into thirds.

Clinician: Oh. Uh-huh.

Marcia: So one-third [points to the circle with the top third shaded] and then three-fourths [points to the circle with the X and three sections shaded], more is shaded in with three-fourths and four-sixths is the same as two-thirds [points to the fractions 4/6 and 2/3 on the pink paper] so ...

Clinician: Mmm-hmm. All right, and so what is the largest fraction in this group?

Marcia: Um, I'd say three-fourths [points to the circle with the X and three sections shaded].

Clinician: Mmm-hmm, and why, and why is that?

Marcia: Because like we knew that one-third is the, the smallest...

Clinician: Mmm-hmm. Mmm-hmm.

Marcia: ...and, um, well, um, one-half [points to the fraction with the left half shaded] is smaller than three-fourths [points to the circle with the X and three sections shaded], 'cause two-fourths is a half...

Clinician: Mmm-hmm. Right.

Marcia: ...and um, well, um two-thirds is the same as that [points to the fraction 4/6 on the pink paper] and that's [points to the fraction 2/3 on the pink paper] smaller than three-fourths.

Clinician: Maybe you could explain to me again why two-thirds is the same as four-sixths.

Marcia: Because, like, if to make four-sixths a lower fraction, um, you could like divide it by two [refers to the fractions 2/3 and 4/6 which she wrote on paper]...

Clinician: Mmm-hmm.

Marcia: and four divided by two; four divided by two is two and six divided by two is three [points to the fractions as she explains].

Comparison #3: In interview #2, when Marcia is asked how she would take one-half of twelve apples she immediately suggests counting the apples and explains that six is half of twelve and six plus six equals twelve, you would take six apples. When asked to show what she means using some of the materials on the table, she arranges twelve red checkers into two rows of six; then she takes six checkers away by pushing the top row into a group. When Marcia is asked how she would take one-third of twelve apples she explains that four times three is twelve, so four is one-third of twelve. As she speaks she pushes four checkers to the right, and then separates the remaining eight into two groups of four, maintaining the row and column configuration. Marcia's external representations here suggest memory images of notation, and additive and multiplicative structures for part-whole relation in a "set" model. She focuses on the numbers of apples, and performs some mental computation. Importantly, her physical manipulation of the checkers suggests an internal pattern image that includes operational one-to-one correspondence between the items in parallel rows.

In contrast, in interview #5 when Marcia is asked to cut a cake into twelve equal pieces, and then to determine the number of pieces with icing on exactly two sides and the top, she initially responds verbally, kinesthetically, and with a drawing. Her representations suggest internal visual and kinesthetic pattern imagery based on linear and region models. She appears to use a "trial and evaluate" halving strategy. Note how her process of measurement appears to give rise to the model she is using here for fractions.

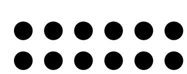
Interview #2

Clinician: Okay. Suppose you had twelve apples, how would you take one-half?

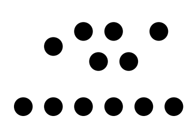
Marcia: Count 'em. Well, if you know there's twelve, and six and since six and six is twelve, take six of them if you want half.

Clinician: Could you show me using something here?

Marcia: All right. [Marcia arranges twelve red checkers into two rows of six and recounts them.]



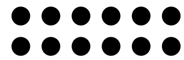
If you know there are twelve apples, you could, and you know that six and six is



twelve you could just take half of them away [pushes six checkers into a pile] and you would have half.

Clinician: Okay, and that's half because, ummm, well because why, why is it one-half?

Marcia: Because, well, if you places it like this and you add the twelve to your six here



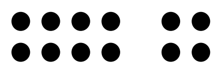
[rearranges the checkers into two rows of six].

Clinician: Yeah.

Marcia: And one, two, three, four, five, six [counts the checkers that were placed back into a straight row], so it's the same amount the other way.

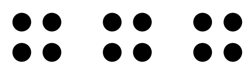
Clinician: All right. And suppose you had the same twelve apples how would you take one third?

Marcia: Ummm, if you knew four times three was twelve, you could take four away and



and you would know it was a third ...

Because four and four is eight and then another four is twelve.



Interview #5

Clinician: All right, let's imagine a cake now. Imagine a birthday cake shaped like a rectangle.

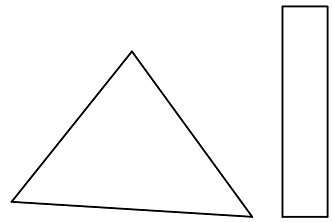
Marcia: [nods]

Clinician: Okay? Imagine what it looks like? Um, can you imagine it?

Marcia: [nods]

Clinician: Okay. What does it look like?

Marcia: Well, um, it's like this [tries to indicate with her hands how the cake would look], like, a, uh, [draws a triangle then says:] oh a rectangle [draws a long thin vertical rectangle].



Clinician: A rectangle. Yeah. Yeah, I want to make sure you're imagining. All right, now imagine that there are twelve people coming to a birthday party and they each want a piece of cake...

Marcia: [nods]

Clinician: ...and your job is to cut the cake so that each person gets the same size piece.

Marcia: [nods]

Clinician: How would you cut the birthday cake?

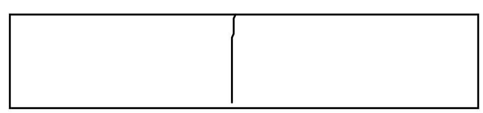
Marcia: Um, well, I think you would have to measure it again like, but it could be, you could make it ...

Clinician: Well, you could just sketch it, if you like.

Marcia: ... and like you would divide the four into half [measures and draws a horizontal rectangle that is four inches long]...

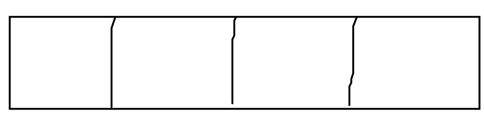
Clinician: Mmm-hmm.

Marcia: ... which is at two and then you would divide each of these into halves ...



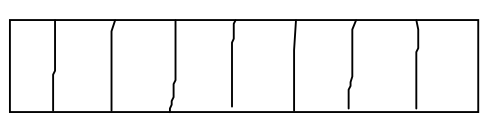
Clinician: Mmm-hmm.

Marcia: ... so you'd have four pieces ...



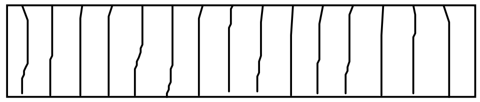
Clinician: Mmm-hmm.

Marcia: ...and then you divide these, which is at the half an inch, 'cause the other one was at the inch to divide...



Clinician: Okay.

Marcia: ... and then you have to divide these which is at an fourth of an inch [counts sixteen pieces].



Clinician: Mmm-hmm.

Marcia: But there's more than twelve so ...

Clinician: Okay, why don't you just draw a sketch while you think of how you would cut the cake into twelve pieces.

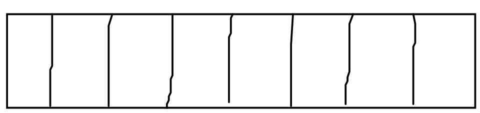
Marcia: Um, well, it would depend on like how big the cake is cause like if the cake is, like, um, twenty-four inches long then you would have to divide them each into two inches.

Clinician: Right.

Marcia: So, it depends on how long the cake is.

Clinician: Yeah, I understand. Just, you can draw a sketch, you don't have to measure it, just draw a sketch, and, and tell me how you might cut the cake into twelve equal pieces.

Marcia: See, but, if you divided these into half there would be sixteen [draws a horizontal rectangle and divides it in half; then divides each half into half, fourths; divides each fourth in half, eighths] ...



Clinician: Mmm-hmm.

Marcia: ...since there's eight now...

Clinician: Yeah.

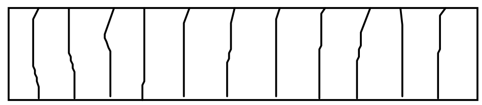
Marcia: ...so it would double it so like you...

Clinician: Yeah, could you do it....

Marcia: ...would have to divide the cake into thirds.

Clinician: Ah-hah.

Marcia: ...and then divide these so then there's six and then you divide these into half which would make that twelve pieces.



Clinician: Okay, so that would be one way to divide the cake ...

Marcia: [nods]

Clinician: ... into twelve equal, would there be any other way to cut the cake into twelve equal pieces, besides you know.

Marcia: [shakes head]

Clinician: No, it would have to be slices like that?

Marcia: [nods]

Clinician: Okay. Okay um now imagine that you put icing on this cake and the icing is on the top ...

Marcia: [nods]

Clinician: ... and around the sides ...

Marcia: [nods]

Clinician: ... on the outsides, hmm.

Marcia: [nods]

Clinician: All right? Umm, now ah, where would the icing be on some of these pieces?

Marcia: Well, for the end pieces and only for the end pieces, it would be on the sides on these sides, well for all of them it would be on these sides [points to horizontal sides]...

Clinician: Yeah.

Marcia: ... and on the top ...

Clinician: Yeah.

Marcia: ... but only for the end pieces, the icing would be on, well, on, on this one it would be on the right side, and on this one it would be on the left side, if you're looking at it like this.

Clinician: Okay, all right. So wha-, how many of those pieces would have icing on the top and just two sides and exactly two sides?

Marcia: Um, there would be ten, 'cause there, it would be on the top here [points to the area inside the rectangle] and then it would be on this side and this side [points to the top and bottom horizontal sides], but you can't count these sides [points to the right and left vertical sides] cause there would be three since there's the edges.

Clinician: All right, um, and what fraction of the pieces then would have icing on the top and two sides?

Marcia: Ten out of twelve, or five-sixths.

Clinician: And where did you get that from?

Marcia: Um, 'cause like there's ten pieces that would have the uh icing only on the top and two sides, so that's ten out of twelve, and then you divide the ten and the twelve by two and you get five-sixths.

LIMITATIONS, COMPARISONS, AND CONCLUSIONS

It is difficult to conjecture broad possible generalizations from comparisons of children in such a small exploratory study. What a child spontaneously expresses, or what she expresses when prompted during an interview at any given moment, may partly be retrieval of images that have most recently occurred in the classroom, at home, or at play. Expression of some particular imagery, or type of imagery, *does not necessarily mean the child's imagery is limited to what has been expressed.* Thus the data are *context-dependent* and *partial*. As researchers, we seek to know and report the context to the maximum extent possible. However, it is impossible for us to know all the individual circumstances. Our inferences are based solely on the children's actual responses during the task-based interviews, in the context of the questions asked and materials available to them during each interview.

The children's internal imagery with respect to fractions is highly varied. We infer a close relation between this imagery and their conceptual development of fraction, based not only on the excerpts presented here, but on our analyses of the full interviews.

Londa and Marcia both used pictorial imagery, pattern imagery, and memory images of symbolic notation when answering the questions, during both interview #2 and interview #5. Inferences of pictorial imagery and pattern imagery were made most frequently, followed by inferences of memory images of symbolic notation, for both children during interview #2; whereas during interview #5, the two children exhibited different levels of frequency for the different characteristics of visual imagery. For Londa in interview #5, inferences of pictorial imagery and pattern imagery were made most frequently, followed by inferences of memory images of symbolic notation. For Marcia in interview #5, inferences of memory images of symbolic notation were made most frequently, followed by inferences of pictorial imagery and pattern imagery.

Londa's imagery is predominantly pictorial, centered in familiar concrete objects. She consistently connects her memory images of symbolic notation with a part/whole conceptual model for fractions, relating each problem directly to a real-life embodiment. Her use of pattern imagery, which might help her abstract and connect different ideas or metaphors of fraction, is not yet well developed. Londa's imagery is especially realistic and personal, as she imagines that the only way to cut a rectangular birthday cake into twelve equal sized pieces is to use a two-row by six-column pattern, the way her own birthday cake was cut. In contrast, Marcia evidences pattern imagery connected with memory images of symbolic notation. She uses part/whole, operator, additive, multiplicative, and linear conceptual models for fractions, and is not "stuck" in particular contexts. Her frequent pattern imagery, and her wider range of conceptual models, suggest that for her symbolic notation is *more flexibly representational* of her imagery. Both children show consistent use of visual imagery, and consistent visual imagery characteristics from the first interview to the second, with evidence of developing memory images of symbolic notation and school-taught algorithms between the two interviews.

But both children evidence a wider range of visual imagery characteristics in interview #2 than in interview #5, including dynamic imagery, and mental operations on images and/or transformed images. The creativity of the children's pictorial imagery seems to diminish from interview #2 to interview #5, even as the frequency and flexibility of the children's pattern imagery increases. The fraction models that predominate in interview #2, and the use of the fraction as an operator, are closely connected to rich imagery drawn from real-life contexts. The models that predominate in interview #5 are more often pattern-oriented or symbolically oriented. The most pronounced change is the development of memory images for symbolic notation in both of the children (used minimally in interview #2 and extensively and comfortably in interview #5), as is to be expected from the children's school experiences.

Our experience with the methodology suggests a final comment. The children's initial verbal responses to questions usually allowed us to infer some imagery. However, the planned prompts for additional external representational forms yielded *considerably more* evidence of *different* internal imagery, *not easily discernible* from the initial, purely verbal responses. Throughout the interviews Londa's and Marcia's drawn pictures and physical manipulations of concrete materials suggested concrete/pictorial and/or pattern imagery; their hand gestures and body movements suggested dynamic imagery and/or mental operations; and their notations suggested memory images of symbols and algorithmic operations. It is essential for the serious study of internal, imagistic representation to provide varied external representational contexts as opportunities for the child's mathematical expressions and productions.

Acknowledgments

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