The Problem of Misperception in Mathematical Visualisation

John Malone & Daniel Boase-Jelinek
Curtin University of Technology

Martin Lamb & Sam Leong
University of Western Australia

Abstract

In a study of 720 Australian school students, some 40% misperceived on very simple mathematical tasks. A second administration of the tasks revealed that 10% (of the total) misperceived on both occasions, so one-on-one interventions took place with this 10% using manipulatives and a specially developed computer program. A further test then showed that a third of the 10% stopped misperceiving whereas another third still misperceived. The study revealed that misperceiving students can be identified and that some misperceptions can be overcome – this latter process provides some illuminating observations for discussion items in Topic Study Group 16.

It is our contention that misperceptions seriously affect students' efforts to learn. We have found that misperceptions affect not only school students, but also pre-service and practicing teachers (Lamb et al, 2002). Teachers who perceive in their mind's eye something that is different to reality are likely to mis-teach. Students who misperceive will fail to see what the teacher is presenting (be it correct or incorrect) and hence cannot learn properly.

The authors have described elsewhere our ongoing study of misperceptions in mathematics and music (Malone, Leong & Lamb, 2001; Lamb, Leong & Malone, 2002). The term *misperception* is defined as the act of perceiving via a single sensory modality (e.g. seeing in mathematics or hearing in music) something that is different from reality or an imagined reality (e.g. visualising a rotated shape in maths). We distinguish it from a *misconception,* which we define as a well established but incorrect belief (Perso, 1991). To date, the study has demonstrated that misperceptions do indeed exist in mathematics, and we have produced a set of examples of typical student misperceptions in order to assist teachers in identifying them. This will result in a more efficient and effective mathematics learning environment.

The mathematics topic selected to be the vehicle for the study was linear transformations, a topic that lends itself conveniently to displaying the misperception phenomenon (Kuchemann, 1982; Sherris, 1999). The negative effects of misperceptions on learning have been largely unappreciated, and learning problems are probably misdiagnosed to be the result of other student errors or misconceptions (Kay & Yeo 2003). On the other hand Shaw, Durden & Baker (1998) describe a student who perceives a 90° angle correctly when aligned horizontally and vertically, but perceives it as 140° when tilted. In other scientific research, positrons were misperceived in data collected over many years, resulting in the delayed identification of a new elementary particle (Solomey, 1998).

In the first (paper-and-pencil) stage of our study we used a flag or a bird for the students to transform in their mind's eye. Figure 1 shows the digitised responses of the entire sample of 720 students to the task of reflecting the flag on the left in the angled mirror. The horizontal flag on the right hand side represents the correct solution. The majority of responses are clearly disappointing as all students were well acquainted with the meaning of the task.

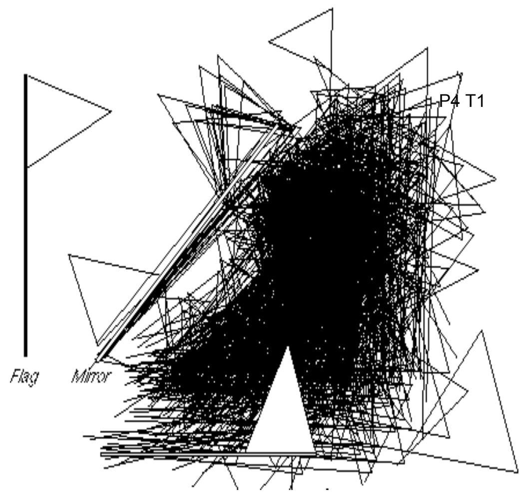


Figure 1

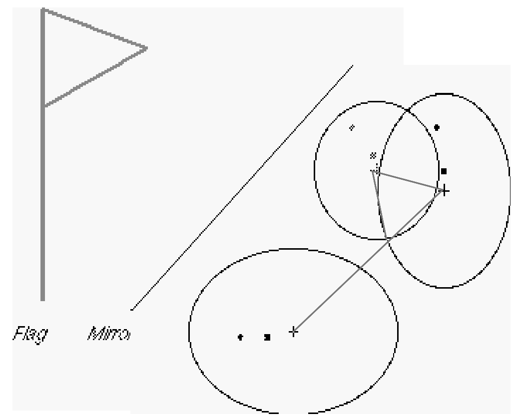


Figure 2

Figure 2 shows an analysis of the responses shown in Figure 1, with the +, □ and * respectively showing the mean, median and modal positions of the base point of the flagpole, the tip of the flagpole and tip of the pennant. Ellipses represent 1 standard deviation from the means. Students were considered to be misperceiving if they could perform a difficult task using the bird, but were unsuccessful in completing a simpler, identical task using the flag. This was to ensure that misperceivers were not confused with misconceivers.

Some weeks later, following a similar set of tests, those students who misperceived consistently were given several one-on-one interventions. The first of these involved, among other things, handing a mirror to each student after getting the task of Figure 1 wrong, to let them check. Their gasps of amazement were indicative of the cognitive conflict taking place. Similar activities (and gasps) took place when using a rotogram following mistakes in rotation tasks. The next intervention involved the use of the computer, as is described next.

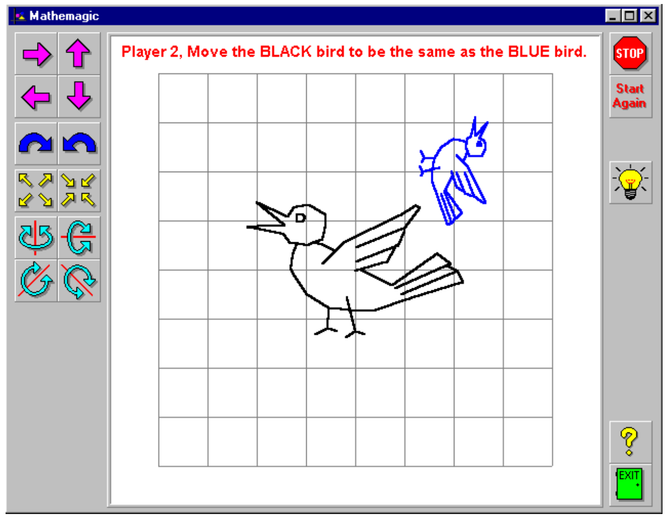
The instrument identified to possess the required functionality to detect and correct misperceptions is called *Mathemagic* (Lamb, 1995) – a title designed to promote interest among school students. By providing an interactive environment that is instantaneously responsive, this software has been able to help us identify misperceptions demonstrated in the workings of individual learners. The *Mathemagic* software allows students to explore the spatial transformations of translation, rotation, dilatation and reflection. Each operation is shown as a slow animation. Transformations can be applied individually or in combination, and the software provides opportunities for problem-solving using the following operations:

(a) Translation (sliding) <u>up</u> or <u>down</u> by one unit; <u>left</u> or <u>right</u> by one unit.
(b) Rotation through a quarter turn (90°) clockwise or anti-clockwise.
(c) Dilatation (scaling) by a factor of 2 or 1/2.
(d) Reflection in the x- or y-axis, or in the line $y = x$, or in the line $y = -x$.

A typical transformation task set for problem-solving is presented as follows: the original picture (in black) in the centre of the screen (see Fig. 3), is shown together with the end result (in blue) of several transformations. The challenge for the student is to find a combination of operations that turns the (original) black picture into the (transformed) blue one.

Figure 3

The solution may be entered in one of two ways:
(1) each of the student's moves may be shown when it is selected, or (2) the moves may only be viewed <u>after</u> all the moves have been chosen, thus requiring the student to visualize the entire sequence of transformations.
The software also has a feature that challenges the student to find the most efficient solution.



The remainder of this paper addresses four of the discussion items of interest to TSG 16. Our responses are based on the findings of the one-on-one interventions of our study.

Discussion Item 3a: What do studies of cognition and diagrammatic reasoning tell us about visual representation in the human brain?

Linear reasoning and logic are processed in the left side of the brain for most right-handed males, while the right side looks holistically at the "big picture" and processes graphical impressions. (For females or left-handers this orientation may be reversed.) However, the corpus callosum ensures that no thought process takes place in just one hemisphere. When students are finding a series of linear transformations to match a randomly generated one, they clearly must use both hemispheres – both logic and pictures are involved. However, the relative time spent in each hemisphere will affect the processing. For example, a strong left hemisphere bias might produce a solution heavily dependent on a point-to-point analysis using calculations; whereas a strong right hemisphere bias might produce a solution based on lateral impressions of the final image – a "gut level" response.

Discussion Item 3b: How can we teach and learn to use visualisation more effectively?

In our experience, strongly left hemisphere-biased approaches tended to be correct. The strongly right bias had more room for misperceptions as it is based on visual impressions. This is akin to the "Grasshopper" (intuitive leap) approach of Dyslexics (Kay & Yeo, 2003). When the latter students had successfully identified the kinds of transformation needed to match the computer's display, they often did not know where on the screen they would end up. A suggestion to "look at where the centre of the image is after each operation" seemed to re-focus them in their left hemisphere where they could complete the solution successfully.

Discussion Item 5: "How does visualisation relate to other ingredients of mathematical understanding," such as orthogonal sets of operations?

In *Mathemagic*, the vertical and horizontal arrows to effect translations are coloured magenta (see Figure 3); the clockwise and anticlockwise curved arrows to effect rotations are coloured blue; and the expanding and contracting arrows to effect dilatation are coloured yellow. The different coloured operations form mutually orthogonal sets: no combination of dilatations will ever make a rotation. After a student has applied a sequence of transformations to a picture on the screen, it is useful for the teacher to examine the sequence together with the student. Looking just at the magenta (translation) operations, if a left arrow and a right arrow occur in different places, they can both be omitted from the sequence without changing the final result – they cancel each other out. Similarly, by examining just the blue (rotation) operations, if both a clockwise and an anticlockwise arrow occur in different places, they can both be omitted from the sequence without affecting the outcome. Clearly, with these three orthogonal sets of operations, any sequence of transformations can be simplified to a shortest solution. (In this way, the concepts of Inverse, Identity and Closure can be demonstrated in a graphical manner.) In our one-on-one interventions, we showed this shortest solution to the student after each task had been completed. Students' reactions to a shorter solution varied from incredulity to an acknowledgement of their "error(s)" and a desire to try again.

Discussion Item 6: What are some of the most effective technology-based tools…..?
We use *Mathemagic* because students can watch transformations actually happening in slow motion. All students believe that the software is "telling the truth". This contrasts with some Physics simulations (Yeo et al, 1999) in which student disbelief was manifest. In front of a class, teachers can demonstrate rotation, reflection and translation using a rotogram and a mirror. While it is more difficult is to demonstrate a dilatation, it is impossible to demonstrate nonlinear transformations such as reflection in a circle without the use of a computer. (In *Mathemagic,* non-linear transformations are provided – our misperceptions study used only the linear transformations). The animation clearly shows how points outside the circle move inside it, while those near the centre of the circle move off the screen ("to infinity"). Repeating the transformation demonstrates that this operation is its own inverse. By using animated sequences of non-linear operations it is possible for students to produce aerofoil-like shapes and *understand* the process by visualizing it.

Conclusions

Our study has revealed disconcerting numbers of students who misperceive linear transformations. By one-on-one interventions using mirrors, rotograms, and a computer program designed to help with visualization, a third of the persistent misperceivers now seem to have overcome their problems with transformations. Our next stage is to work one-on-one with the similar number of recalcitrant misperceivers and help them to compose music using linear transformations, with another computer program *Musicland*. We will then observe how this cross-pollination affects their mathematical misperceptions.

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