

# Visual representations in the construction of mathematical meanings<sup>1</sup>

Michela Maschietto<sup>2</sup>, Maria G. Bartolini Bussi<sup>2</sup>

Maria Alessandra Mariotti<sup>3</sup>, Franca Ferri<sup>4</sup>

## Introduction

The impact on the teaching and learning processes of the use of material instruments is the focus of the educational research carried out by our group. The parallel issue concerning the use of ‘virtual’ objects (also much cheaper and easier to distribute to distant locations) included in multimedia is now being studied by educators, who already have some results on the condition of functioning in the classroom. A special issue of *Learning and Instruction* 13 (2003, 117-254) has been devoted to this question. In the introduction of the topic, the editors Schnotz and Lowe emphasise that multimedia resources can be considered at three levels: the technological level (i.e. software features), the semiotic level (referred to the representation format) and the sensory level (i.e. visual or auditory modality). With regard to these three levels, the authors have found some misconceptions among educators because of a failure to distinguish them. One of these misconceptions is that the technical medium is only level to be considered and it is presumed to have an impact on learning. The authors suggest that:

*“Rather than searching for technical media-effects, research on learning and instruction should focus on the semiotic and sensory levels of multimedia. The emphasis should be on the effects of different forms of external representation such as texts and graphics (either static or animated) on comprehension and learning. Unfortunately, there are also misconceptions with regard to multimedia's semiotic and sensory levels. A widespread misconception of this type is the assumption that rich multimedia learning environments result in extensive cognitive processing and thus create elaborated knowledge structures. (...). The resulting learning materials include a diverse range of components including animation, videoclips and various possibilities for interaction. However recent research indicates that multiple external representations and multiple modalities are not always beneficial for learning. For this reason, a better understanding of the processing demands associated with different kinds of representation and their function in comprehension and learning is required. It is becoming increasingly apparent that during multimedia learning, complex interactions take place between the external representations provided by increasingly sophisticated educational technologies and the internal (mental) representations constructed by learners whose instruction is presented via those technologies. The potential effectiveness of educational multimedia materials is therefore likely to be influenced by the extent to which their design takes these complex interactions into account.” [117-118]*

In this paper we only intend to discuss the following issue: how can students make sense of interactive material that is offered to them ready to be explored? If the aim is the teaching and learning of mathematics (i. e. the construction of mathematical meaning, the practice of mathematical processes, the production of conjectures and proofs) it is essential to compare and contrast the exploration of real and virtual copies, and discuss different kinds of animations (sketches, semi-realistic and photorealistic ones) produced using different tools. We focus on instruments for perspective drawings and their photorealistic animations (produced by Cinema4d). In this case the features of the software may even be unknown to the students who rather explore a virtual world that recalls the videogame settings. In general, these animations are not easy to interpret, as we proved several times with adult.

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<sup>2</sup> Dept of Mathematics, University of Modena and Reggio Emilia (Italy)

<sup>3</sup> Dept. of Mathematics, University of Pisa (Italy)

<sup>4</sup> Scuola Elementare Palestrina, Modena (Italy)

## Theoretical framework

The experiment considered in this paper concerns the perspective drawing, which is a field of experience that allows pupils to construct a germ-theory within which they may produce examples of theorems. It is related to the Desarguesian approach to geometry. Similar experiments had been replicated several times (Bartolini Bussi, 1996), but this one was enriched with the introduction of a large Dürer's glass in the classroom. The analysis of this object's role was carried out (Bartolini Bussi and al., to appear), with reference to the works of Vygotsky and Bachtin, combined with recent contributions from activity theorists (Ergostroem, Wartofsky) and the transposition of the theoretical construction of semiotic mediation into educational design and classroom implementation. They deepen the study of relations between the technical and psychological tools (which are not distinguished in Vygotsky) with respect to cultural artefacts, i.e. primary, secondary and tertiary artefacts. Primary artefacts are "*those directly used*" and secondary artefacts are "*those used in the preservation and transmission of the acquired skills or modes of action*". According to Wartofsky, technical tools correspond to primary artefact whereas psychological tools to secondary artefacts. The tertiary artefacts usually correspond to objects described by rules and conventions and that are not strictly connected to practice. For example, mathematical theories are tertiary artefacts, within which the models constructed as secondary artefacts are organised. The appropriation of tertiary artefacts may be described as the construction of mathematical meanings and ways of thinking. With an educational perspective, uses of primary and secondary artefacts may allow pupils to reconstruct the origins of some mathematical concepts that those artefacts refer to. The experiment was designed according to all these assumptions.

## The teaching experiment

The experiment was designed for 4<sup>th</sup> – 5<sup>th</sup> grade classes. In accordance with the theoretical background, it was composed of several steps described below. We specify that steps 1 and 2 were carried out at the end of the 4<sup>th</sup> grade class, while steps 3-15 constituted a part of the mathematical curriculum of the 5<sup>th</sup> grade class.

1. Exploration of a primary artefact. A Dürer's was introduced in the classroom. It shows the skeleton of a cube and its perspective drawing. Dürer's glass is the simplest perspectograph and is composed of an eyehole and a transparent screen where the artist traces the apparent contour of the object directly. Pupils discovered it during a mathematical discussion (Bartolini Bussi, 1996). They could use this instrument.
2. Individual drawings of the primary artefact.
3. Interpretation of secondary artefacts. During a discussion, pupils were asked to interpret some excerpts of texts drawn from ancient treatises on paintings (Piero della Francesca, L.B. Alberti).
4. Construction of a table of invariants in the shift from reality to representation (Bartolini Bussi, 1996).
5. Focus on the transformation of rectangular shapes and construction of the definitions of several types of quadrilaterals.
6. Design of a 'new' tool for perspective drawings (effective real life drawings): pupils worked in small groups; they then had to present their tool to the class (spokesman's presentation).
7. Individual text on the project, where each pupil presented the projected tool.
8. Individual text on Alberti's visual pyramid: L.B. Alberti's sentence on the visual pyramid, which was discussed previously in step 3, is proposed for an individual task. We will analyse some excerpts of protocols in this paper.
9. Exploration and use of a new primary artefact. A model for the visual pyramid was introduced in the classroom. A discussion followed.
10. Individual text on the artefact. As was planned at the beginning of the experiment, an individual written production, where pupils presented their interpretations of the considered model, followed the work on the primary artefact. The texts contain descriptions and drawings.

11. Interpretation of a PC simulation of known and unknown tools. PC simulations of some perspectographs were shown to pupils: there were simulations of Dürer's glass and of a perspectograph unknown to pupils.
12. Individual text on the PC simulations. Each pupil had chosen one of the PC simulations and received one printed frame by the teacher. The pupils then wrote the descriptions of the chosen simulations. We will analyse some protocols in this paper.
13. Visit to the *Perspectiva Artificialis* public exhibition, which took place in Modena in May 2003 (for a virtual visit, go to [www.mmlab.unimore.it](http://www.mmlab.unimore.it) and choose <Perspectiva Artificialis>).
14. Individual text on the visit. Pupils were asked to describe an instrument they had seen at the exhibition.
15. The pupils presented their experience on perspectograph on a national newspaper on line.

### Some results on artefacts

Some results from the analysis of the experiment up to step 3 are presented in (Bartolini Bussi et al., to appear). The first two steps are centred on primary and secondary artefacts, whereas the following steps concern the pupils' appropriation of styles of mathematical thinking. At the beginning of the discussion, with respect to step 3, the sentence that introduces the first mathematical model for Dürer's glass, which is no longer available in the classroom, is "*Thus painting will be nothing more than intersection of the visual pyramid*"<sup>5</sup> by L.B. Alberti (*De Pictura*, 1540). The two key words that lead to the pupils' interpretation of the sentence are 'visual pyramid' and 'intersection'. The word 'pyramid' seems to be easy to interpret, with the help of the reference to Egyptian monuments. The word 'intersecazione' (the ancient spelling for 'intersezione', i.e. intersection) is interpreted by an example from an everyday word: it is related to the word 'segare' (i.e. to saw). A dialogue takes place between the concrete referent and the ideal model, as the following excerpt shows:

**Alessandro B.:** If the base is triangular it has 4 [faces], if the base is square it necessarily has 5. It depends on the base. The one we are talking about has either a square or a rectangular base, because we imagine a painting or a piece of glass and the point of the triangles reaches the eye.

**Federica:** Yes, but Leon Battista Alberti's is not a real solid, it's an imaginary solid which takes shape while you're looking at it. We can't see it, we can see it only when we think of it, if we want to see it. For example we can see it now because we have just read it.

**Assia:** Of course it's imaginary, otherwise it would harm you and then it wouldn't even allow you to see.

**Voices:** Can you imagine a solid getting into your eye!

*[Many gestures, funny ones as well! A moment of confusion and jokes about the visual pyramid with participation of the entire class].*

The words are not the only signs involved in this complex semiotic activity. Gestures and drawing are very important: gestures mime planes and lines and constitute a fundamental support to imagine a pyramid, while the teacher's drawing, requested by a pupil, seems to support pupils' mental imaging.

**Federica:** Yes, all right, but in any case you have to imagine it. I understood this, if you saw it near the object you obtain a large image, if you saw it near the eye you get a smaller image.

*[With gestures, many children cut, saw the visual image. They trace many imaginary planes which are parallel to the painting.]*

**Alessandro B.:** If you go down straight, because with our hands we form a kind of plane parallel to the one of the objects *[With his hands he traces two parallel planes in space]*. In this way you certainly obtain a figure which is exactly the same as the base of the pyramid, but smaller.

This first phase of the experiment shows the importance of a direct link with the experience of a primary artefact and the key role of text (as a secondary artefact). It is given as a sign to be

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<sup>5</sup> "Sarà adunque pittura non altro che intersecazione della piramide visiva"

interpreted and it is used by the teacher as a tool of semiotic mediation in order to make personal meanings evolve towards mathematical meanings. The function of bodily experience and social interaction is also emphasized in this part of the experiment.

### Visual pyramid and animation

As previously mentioned, we essentially focus on steps 8 and 12. For these steps, tasks are quite different. For step 8, pupils were asked to write their interpretations of L.B. Alberti's sentence ("Thus painting will be nothing more than intersection of the visual pyramid") three months after the discussion of step 3. For step 12, they had to interpret some photorealistic animations, without any verbal explanation or adult guidance.

For step 8, analysis of the pupils' protocols highlights that this task allows them to express their internal representation of the visual pyramid and the semiotic game, within which meanings may be evoked and then evolve. In these protocols, fundamental elements of mathematical model may appear: only one eye must be used, visual rays which constitute visual pyramid are straight, relationships between the size of perspective images of an object and distance between this object and the eye are established, the invariance of shapes of images when visual pyramid is cut by parallel planes is understood. In pupils' protocols, gestures that mimed planes and lines during the discussion of step 3 became very precise signs connected to the word 'intersection' and support the explanation of L.B. Alberti's sentence.

The protocols below contain the described elements; in particular, figure 2 presents the parallel planes that intersect the visual pyramid, with the change in size of drawn objects.

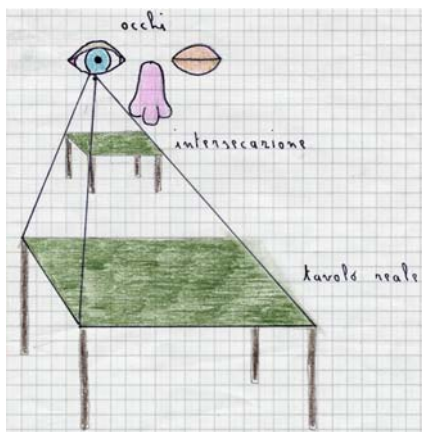


Figure 1 – Alessandro B.



Figure 2 – Elisabetta

In Assia's protocols, after stressing the origin of visual pyramid in the iris, she tried to explain the nature of the visual pyramid, which shows that the ambiguity between concrete referent (a pyramid) and ideal object was maintained with respect to previous discussion. That is a rich ambiguity. Assia wrote: "I want to say the eye imagines a transparent pyramid, that is a visual one that is not seen". In her text, the word 'visual' is connected to the eye that the pyramid arises from, but the object of the seeing action is not a material one. In Giacinto's text (figure 3), the fundamental elements were: the necessity of only using one eye to see a perspective image, the necessity of the drawn (and also named 'intersecazione') intersection to obtain a visual pyramid and the fact that this intersection is equivalent to the "sawing" action. In this intersection, human eye sees the obtained image, whose magnitude is related to the distance between the intersection and the human eye. In his drawing, Giacinto gave an explicit reference to a perspectograph and so to his experience with real instrument, because he labelled the open eye as 'eyehole'. In his protocol (figure 4), Lorenzo described two situations concerning the visual pyramid: they were characterised by the presence or absence of the intersection. He emphasised that the 'intersecazione' breaks the visual pyramid and

obtained a perspective image only in this case. Otherwise, a person sees an object without the need to draw it.

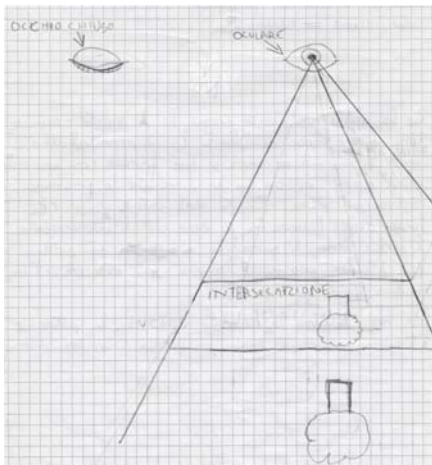


Figure 3 – Giacinto

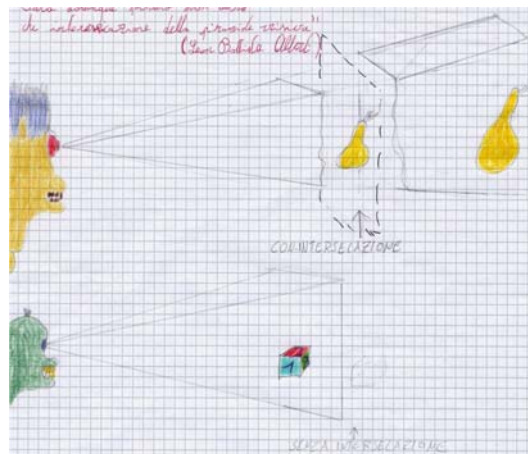


Figure 4 – Lorenzo

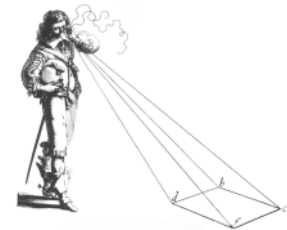
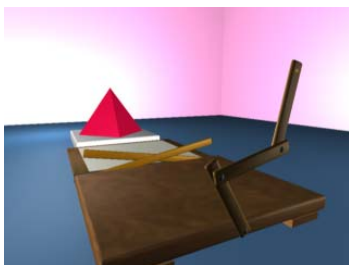


Figure 5

In most of the protocols, we can distinguish two parts: a first part including the verbal description of the visual pyramid (internal representation) and a second part where pupils comment L.B. Alberti's sentence. We may consider that the pupils made a re-interpretation of the said sentence, which they had already discussed. In their protocols, there were different (text and graphics) and rich forms of external representations, which the pupils used in a coherent and complementary way. It is interesting to compare these representations with a historical one (figure 5, by Abraham Bosse in *Manière Universelle de M. Desargues*), that shows the same ambiguity between real and ideal features of a visual pyramid. With respect to Wartosky's distinction on artefacts, pupils produced secondary artefacts at this point, because they explained acquired action modes, supported by their constructed mathematical meanings.

For step 12, where the pupils had to describe a chosen PC simulation ("*Describe the machine represented in the image. You can use meaningful drawings*"), the analysis of pupils' protocols highlighted how animations of new instruments may be understood when the pupils are already familiar with similar instruments and have discussed the mathematical model that is behind them, as appears in the protocols of step 8. The proposed animations concerned perspectographs that had features of the instruments already used in the classroom.

Here is an excerpt of Giacinto's text, where the pupil explains the functioning of the perspectograph (figure 6) identifying the intersection (he writes 'intersecazione') of the visual pyramid on the sheet on the mobile plane.



Danti's window is used for drawing in perspective plane and solid geometrical figures.

The functioning of the machine is: you have to put the eye in the eyehole, then you have to see the vertex that corresponds to the bar crossing. Then lift the plane, make a point in the bar crossing. And you make the same for all the vertices you can draw. At the end join the points and the figure you wished to draw comes up. I think that the sheet is the 'intersecazione' (intersection).

Figure 6 – An animation frame

The visual pyramid, in particular the 'intersecazione', is also evoked by Lorenzo (figure 8). In his protocol, it is possible to observe a correspondence between the interpretation of animation and that of the L.B. Alberti's sentence. With respect to the drawing interpreting the L.B. Alberti's sentence (figure 4), the plane (of the pyramid with 'intersecazione') is materialised on the sheet of the chosen perspectograph (an original version of Dürer's glass, figure 7). In this case, the sheet allows the

intersection, which is necessary to obtain the perspective image of the cube. Besides, this protocol is an example of the previously discussed ambiguity between the concrete referent and the ideal one maintained by the animations: Lorenzo considered the visual rays starting from the eyehole as imaginary threads connected to the vertex of the cube.

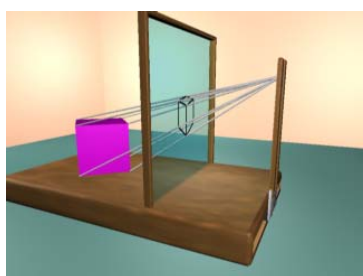


Figure 7 – An animation frame

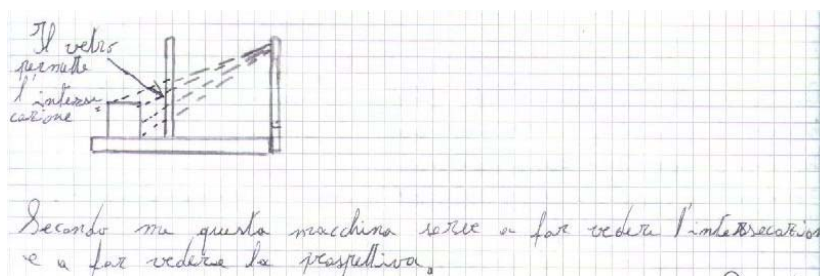


Figure 8 – Lorenzo

Photorealistic animations constitute new secondary artefacts to be interpreted by pupils. All the animations present the same mathematical model, even if they have specific features. Most of the pupils chose to describe a new instrument (Abate Lerino's perspectograph or Danti's window) and they evoked and coherently used some elements with their text in step 8. The analysis seems to highlight that the activities of representation carried out in the experiment allow the pupils to make sense to the photorealistic animations.

### Concluding remarks

In this paper we wanted to take into consideration the following issue: how can students make sense of interactive materials that are offered them ready to be explored? This question is studied in the case of a teaching experiment where the use of real copies of instruments goes with photorealistic animations of these instruments. We have considered the role of representations in building mathematical meanings (i.e. a germ-theory of Desarguesian approach to geometry). We have analysed two different tasks where it is required to produce representations: the explanation of a historical sentence and the interpretation of computer animations. For the first task, only the sentence of L.B. Alberti is given: the pupils explained it with words and drawings. These drawings were used to explain meanings of visual pyramid and contained essential elements of mathematical model. We specify that the drawings were not especially mentioned in the task, but the pupils usually used them in their written production. For the second task, the pupils had to comment a chosen computer animation (that is a 'dynamic object' with respect to the sentence); besides, one animation frame is given to each pupil. Also in this task, the pupils traced drawings to support their explanation, which were coherent with the previous representations. In general, these animations are not easy to interpret, as we proved several times with adult. They contain an ambiguity between material threads and visual rays, which is often causing of misunderstanding for people. Our analysis of the pupils' protocols shows how the pupils succeeded in the second task. In particular, mentioned ambiguity is not a source of problem for pupils, but supports their representations of the visual pyramid, which contains similar ambiguity. Their texts about Alberti's sentence emphasise the richness of this ambiguity.

The analysis of our experiment is consistent with the position expressed by Schnotz and Lowe in the introduction: if we want to use animations in teaching and learning mathematics, we have to contribute to the construction of mathematical meaning implicated in animations. We have offered here an example where we tried to build this construction for a Desarguesian approach to geometry in primary school. In our research, specific activities on cultural artefacts made up to the vision of animations seem to allow pupils to construct a tool for the interpretation of a complex artefact. Nevertheless, the question we have considered in this paper remains an important open question.

## Reference

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